UNIT - III

FOURIER SERIES and Fourier Transforms

Fourier Series: 1. Fourier series is an infinite series representation of a periodic functions, in terms of the trigonometric sines & cosines

2. Fourier series is possible for continuous functions, periodic functions & functions discontinuous in their values & derivatives

3. Fourier series is useful to solve partial differential equations with boundary value problems.

Euler's (Fourier-Euler) Formulae.

Definition: Let the function $f(x)$ be periodic of period '2π'. The fourier series of $f(x)$ in the interval $[c, c+2\pi]$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,}$$ where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx, \; n=1,2,3\ldots$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx, \; n=1,2,3,\ldots$$

Here, a_0, a_n, b_n are called the fourier coefficients & the above coefficients are also known as Euler's coefficients.

Note: 1) If $c=0$ then the interval becomes $[0, 2\pi]$ & the constant coefficients are given by $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

2) If $c=-\pi$ then the interval becomes $[-\pi, \pi]$ & the constant coefficients are given by $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Conditions for Fourier Series (or) Dirichlet's conditions:

* A function $f(x)$ has a fourier series expansion of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where a_0, a_n, b_n are constants, provided

1. $f(x)$ is periodic, single valued & finite.
2. $f(x)$ has a finite no. of discontinuities in any one period
3. $f(x)$ has atmost finite no. of maxima & minima in any given interval.

These conditions are known as Dirichlet's conditions.

Important Formulae:

1. $\sin n\pi = \sin(n+1)\pi = \sin 2n\pi = \sin 2(n+1)\pi = 0$, $\forall n \in Z$, where Z is the set of all integers.
2. $\cos n\pi = (-1)^n$, $\cos(n+1)\pi = (-1)^{n+1}$, $\cos 2n\pi = (-1)^{2n}$, $\cos 2(n+1)\pi = (-1)^{2(n+1)}$, $\forall n \in Z$
3. $\sin(n+\frac{1}{2})\pi = (-1)^n$, $\forall n \in Z$
4. $\cos(n+\frac{1}{2})\pi = 0$, $\forall n \in Z$

1. Expand $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$ in a fourier series.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ be the fourier series —①

where a_0, a_n, b_n are the obtained using Euler's formulae

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 dx \qquad \text{ILATE}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\,dx = \frac{-1}{4\pi}\int_0^{2\pi}(\pi-x)^2(-1)\,dx$$

$$= \frac{-1}{4\pi}\left[\frac{(\pi-x)^3}{3}\right]_0^{2\pi} \qquad \left(\because \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}\right)$$

$$= \frac{-1}{12\pi}\left[-\pi^3 - \pi^3\right] = \frac{2\pi^3}{12\pi} = \frac{\pi^2}{6} \Rightarrow \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \cos nx\, dx$$

$f(x) = (\pi - x)^2$, $g(x) = \cos nx$

$f'(x) = -2(\pi - x)$ $\quad \int g(x)\,dx = \frac{\sin nx}{n}$

By integration by parts,

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left(f'(x)\int g(x)\, dx\right)dx$$

$$\therefore a_n = \frac{1}{4\pi}\left[\left((\pi - x)^2 \frac{\sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi}\left(-2(\pi - x)\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{1}{4\pi}\left[\left(\pi^2 \frac{\sin 2n\pi}{n}\right) - \pi^2(0)\right] + \frac{1}{2n\pi}\int_0^{2\pi}(\pi - x)\sin nx\, dx$$

$$= \frac{1}{4\pi}(0 - 0) + \frac{1}{2n\pi}\int_0^{2\pi}(\pi - x)\sin nx\, dx = \frac{1}{2n\pi}\int_0^{2\pi}(\pi - x)\sin nx\, dx$$

$$= \frac{1}{2n\pi}\left[\left((\pi - x)\left(-\frac{\cos nx}{n}\right)\right)_0^{2\pi} - \int_0^{2\pi}(-1)\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{2n\pi}\left[\frac{-1}{n}\left[(-\pi)\underbrace{\cos 2n\pi}_{(1)} - \pi(1)\right]\right] - \frac{1}{2n^2\pi}\left(\frac{\sin nx}{n}\right)_0^{2\pi}$$

$$= \frac{1}{2n\pi}\left[\frac{-1}{n}\left[-\pi - \pi\right]\right] - \frac{1}{2n^3\pi}\left[\sin 2n\pi - 0\right]$$

$$= \frac{+2\pi}{2n^2\pi} - 0 = \frac{1}{n^2} \Rightarrow \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx\, dx = \frac{1}{4\pi}\left[\left((\pi - x)^2\left(-\frac{\cos nx}{n}\right)\right)_0^{2\pi} - \int_0^{2\pi} -2(\pi - x)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{4\pi}\left[\frac{-1}{n}\left[\pi^2 \cos 2n\pi - \pi^2\right]\right] - \frac{1}{2n\pi}\int_0^{2\pi}(\pi - x)\cos nx\, dx$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}(\pi^2 - \pi^2)\right] - \frac{1}{2n\pi}\left[\left((\pi - x)\frac{\sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi}(-1)\frac{\sin nx}{n}\, dx\right]$$

Scanned with CamScanner
Scanned with CamScanner

$= 0 - \frac{1}{2n\pi}\left[\left[\frac{1}{n}(-\pi)\sin 2n\pi - \pi(0)\right] \frac{1}{2n^2\pi} \frac{-\cos nx}{n} - \frac{1}{n^2}(\cos 2n\pi - \cos 0)\right.$

$= \frac{-1}{2n\pi}\left[0 - \frac{1}{n^2}(1-1)\right] = 0 \Rightarrow \boxed{b_n = 0}$

Substituting the values of a_0, a_n, b_n in ①, we get

$$f\left(\left(\frac{\pi - x}{2}\right)^2\right) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty} 0 \sin nx$$

$$\therefore f\left[\left(\frac{\pi - x}{2}\right)^2\right] = \frac{\pi^2}{12} + \cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots$$, is the required fourier series

Imp

Q Obtain fourier series for the function, $f(x) = x\sin x$ in $0 < x < 2\pi$

Sol. given $f(x) = x \sin x$, $0 < x < 2\pi$

~~The fourier~~ Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — ①

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$

$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$

$= \frac{1}{\pi}\left[\left(x(-\cos x)\right)_0^{2\pi} - \int_0^{2\pi}(-\cos x)\,dx\right]$

$= \frac{1}{\pi}\left[\left(2\pi \cos 2\pi - 0\right)\right] + \left(\sin x\right)_0^{2\pi}\Big]$

$= \frac{1}{\pi}\left[-2\pi \cancel{+1}\right] + \frac{1}{\pi}(0-0) = -2 \Rightarrow \boxed{a_0 = -2}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx \sin x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x \sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x \sin(n-1)x\,dx$

$a_n = I_1 - I_2$ — ②

To find I_1: $I_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin(n+1)x\,dx = \frac{1}{2\pi}\left[\left(x\left(\frac{-\cos(n+1)x}{n+1}\right)\right)_0^{2\pi} - \int(1)\left(\frac{-\cos(n+1)x}{n+1}\right)dx\right]$

Scanned with CamScanner

Scanned with CamScanner

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}\left(2\pi \cos 2(n+1)\pi - 0\right)\right] + \frac{1}{2\pi(n+1)}\left(\frac{\sin(n+1)x}{n+1}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}(2\pi)\right] + \frac{1}{2\pi(n+1)^2}\left[\sin(2)(n+1)\pi - 0\right]$$

$$= \frac{-1}{n+1} + \frac{1}{2\pi(n+1)^2}[0-0] = \frac{-1}{n+1} \quad \left(\because \begin{array}{l}\cos 2(n+1)\pi = 1 \\ \sin 2(n+1)\pi = 0\end{array}, \forall n \in Z\right)$$

$$\therefore I_1 = \frac{-1}{n+1}$$

To find I_2: $I_2 = \frac{1}{2\pi}\int_0^{2\pi} x \sin(n-1)x\,dx = \frac{1}{2\pi}\left[\left(x\left(\frac{-\cos(n-1)x}{n-1}\right)\right)_0^{2\pi} - \int_0^{2\pi} 1\left(\frac{-\cos(n-1)x}{n-1}\right)dx\right]$

$$= \frac{1}{2\pi}\left[\frac{-1}{n-1}\left(2\pi \cos 2(n-1)x - 0\right)\right] + \frac{1}{2\pi(n-1)}\left(\frac{\sin(n-1)x}{(n-1)}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n-1}(2\pi - 0)\right] + \frac{1}{2\pi(n-1)^2}\left[\sin 2(n-1)x - 0\right]$$

$$= \frac{-1}{n-1} + \frac{1}{2\pi(n-1)^2}(0-0) = \frac{-1}{n-1}$$

(2) $\Rightarrow a_n = I_1 - I_2 = \frac{-1}{n+1} - \left(\frac{-1}{n-1}\right) = \frac{1}{n-1} - \frac{1}{n+1} = \frac{n+1-n+1}{n^2-1} = \frac{2}{n^2-1}$

$$\therefore \boxed{a_n = \frac{2}{n^2-1}}, \quad \text{if } n \neq 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin nx \sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\,dx$$

$$b_n = I_1 - I_2 \quad \text{--- (3)}$$

To find I_1: $I_1 = \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\,dx = \frac{1}{2\pi}\left[\left(x\,\frac{\sin(n-1)x}{n-1}\right)_0^{2\pi} - \int_0^{2\pi}\frac{\sin(n-1)x}{n-1}dx\right]$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\left(2\pi \sin 2(n-1)\pi - 0\right)\right] - \frac{1}{2\pi(n-1)}\left(\frac{-\cos(n-1)x}{n-1}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}(2\pi(0) - 0)\right] + \frac{1}{2\pi(n-1)^2}\left[\cos 2(n-1)\pi - 1\right]$$

$$= \frac{1}{n-1}(0) + \frac{1}{2\pi(n-1)^2}(\cancel{1} - \cancel{1}) = \frac{1}{n-1}(0) = 0, \text{ if } n \neq 1$$

Scanned with CamScanner

To find I_2: $I_2 = \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\,dx$

$$= \frac{1}{2\pi}\left[\left(x\,\frac{\sin(n+1)x}{n+1}\right)_0^{2\pi} - \int_0^{2\pi}\left(\frac{\sin(n+1)}{n+1}\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}\left(2\pi\sin 2(n+1)x - 0\right)\right] - \frac{1}{2\pi(n+1)}\left(\frac{-\cos(n+1)}{n+1}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}(0-0)\right] + \frac{1}{2\pi(n+1)^2}\left[\cos 2(n+1)\pi - 1\right]$$

$$= 0 + \frac{1}{2\pi(n+1)^2}[01-1] = \underline{0}, \text{ if } n\neq 1$$

③ $\Rightarrow$ $b_n = 0 - 0 = 0 \Rightarrow \boxed{b_n = 0}$, if $n \neq 1$

We got $a_n = \frac{2}{n^2-1}$, if $n\neq 1$

If $n=1$, we have $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cos x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[\left(x\left(\frac{-\cos 2x}{2}\right)\right)_0^{2\pi} - \int_0^{2\pi}\left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{2}(2\pi\cos 4\pi - 0)\right] + \frac{1}{4\pi}\left(\frac{\sin 2x}{2}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{2}(2\pi - 0)\right] + \frac{1}{8\pi}\left[\sin 4\pi - 0\right]$$

$$= \frac{1}{2\pi}(-\pi) + 0 = \underline{\frac{-1}{2}} \Rightarrow \boxed{a_1 = \frac{-1}{2}}$$

~~① $\Rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$~~

We got $b_n = 0$, for $n \neq 1$

$\therefore$ If $n=1$, we have $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\,dx$$

$= \frac{1}{2\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} - \frac{1}{2\pi}\left[\left(x\,\frac{\sin 2x}{2}\right)_0^{2\pi} - \int_0^{2\pi}\frac{\sin 2x}{2}\,dx\right]$

$= \frac{1}{4\pi}(4\pi^2) - \frac{1}{2\pi}\left[\frac{1}{2}(2\pi\sin 4\pi - 0)\right] + \frac{1}{4\pi}\left(-\frac{\cos 2x}{2}\right)_0^{2\pi}$

$\pi - \frac{1}{4\pi}(0-0) + -\frac{1}{8\pi}\left[\cos 4\pi - \cos 0\right] \quad \pi - \frac{1}{8\pi}(1-1) = \pi$

$\Rightarrow \boxed{b_1 = \pi}$

① $\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$= -\frac{2}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$

$= -1 + -\frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}\right)\cos nx + \pi\sin x + \sum_{n=2}^{\infty}(0)\sin nx$

$\therefore f(x\sin x) = -1 - \frac{1}{2}\cos x + \pi\sin x + \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x + \frac{2}{15}\cos 4x + \cdots$, is the required fourier series

Convergence of Fourier Series: If $x=a$ is the point of continuity of $f(x)$ then the fourier series of $f(x)$ is converged to $f(a)$.

If $x=a$ is a point of discontinuity of $f(x)$ then the fourier series of $f(x)$ is converged to $\frac{1}{2}\left[f(a^-) + f(a^+)\right]$,

where $f(a^-)$ = left hand limit, $f(a^+)$ = Right hand limit.

1. Find the fourier series to represent the function, $f(x)$, is given by $f(x) = x, \; 0 \le x \le \pi$ & hence deduce that

$= 2\pi - x, \; \pi < x \le 2\pi$

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol. Given, $f(x) = x, \quad 0 \le x \le \pi$

$= 2\pi - x, \quad \pi < x \le 2\pi$

Fourier series for a period of 2π is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \quad$ —①

where $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$

Scanned with CamScanner

Scanned with CamScanner

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi\,(x)_{\pi}^{2\pi} - \left(\frac{x^2}{2}\right)_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \frac{4\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\pi^2 + 2\pi^2 - 2\pi^2\right] = \underline{\pi} \Rightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\,\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}\left(\frac{\sin nx}{n}\right)dx + \left((2\pi - x)\,\frac{\sin nx}{n}\right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\left(-1\,\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(\pi\sin n\pi - 0) + \frac{1}{n}\left(\frac{\cos nx}{n}\right)_0^{\pi} + \frac{1}{n}(0 - \pi\sin n\pi) - \frac{1}{n}\left(\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= 0 + \frac{1}{n^2\pi}(\cos n\pi - \cos 0) + 0 - \frac{1}{n^2\pi}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1) - \frac{1}{n^2\pi}(1 - \cos n\pi) = \frac{1}{n^2\pi}(\cos n\pi - 1) + \frac{1}{n^2\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \qquad \left(\because \begin{array}{l}\sin n\pi = 0\\ \cos n\pi = (-1)^n\\ \cos 2n\pi = (-1)^{2n} = 1\end{array}\right)$$

$\therefore\ a_n = 0$, for n is even

$= \dfrac{-4}{n^2\pi}$, for n is odd.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\left(\frac{-\cos nx}{n}\right)\right)_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right)dx + \left((2\pi - x)\left(-\frac{\cos nx}{n}\right)\right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi}(-1)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left[\pi\cos n\pi - 0\right] + \frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left(-\frac{1}{n}\right)(0 - \pi\cos n\pi) - \frac{1}{n}\left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{-1}{n}\cos n\pi + \frac{1}{n^2\pi}(\sin n\pi - 0) + \frac{1}{n}\cos n\pi - \frac{1}{n^2\pi}(\sin 2n\pi - \sin n\pi)$$

$$= \frac{1}{n^2\pi}(0 - 0) - \frac{1}{n^2\pi}(0 - 0) = \underline{0} \Rightarrow \boxed{b_n = 0}$$

$$① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} 0\,\sin nx$$

$= \frac{\pi}{2} + \sum_{n=1,3,5..}^{\infty} \frac{-4}{n^2\pi} \cos nx + 0 \quad = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$ — ②

Deduction: At $x=0$, $f(x)$ is continuous.

Hence the fourier series is converged to $f(0)$

put $x=0$ in ②, we get $f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$\Rightarrow \cancel{-}\frac{\pi}{2} = \cancel{-}\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

~~1~~ 15

Q. Find the fourier series of $f(x) = x + x^2$, $-\pi < x < \pi$ & hence deduce the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Sol. given $f(x) = x + x^2$, $-\pi < x < \pi$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— ①}$$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$ $\quad$ ($\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, $f(x)$ is even

$= 0 + \frac{2}{\pi}\int_0^{\pi} x^2\,dx$ $\quad$ $= 0$, $f(x)$ is odd

$f(-x) = -f(x) \to$ odd

$= \frac{2}{3\pi}\left(x^3\right)_0^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$ $\quad$ $f(-x) = f(x) \to$ even)

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\cdot dx$

$= 0 + \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\cdot dx$ $\quad$ ($\because$ $x\cos nx$ is odd function

$x^2 \cos nx$ is even function)

$= \frac{2}{\pi}\left[\left(x^2 \frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}\left((2x)\frac{\sin nx}{n}\right)dx\right]$

$= \frac{2}{\pi}\left[\frac{1}{n}(\pi^2 \sin n\pi - 0)\right] - \frac{4}{n\pi}\int_0^{\pi} x \sin nx\cdot dx$

$$= 0 - \frac{4}{n\pi}\left[\left(x\left(\frac{-\cos nx}{n}\right)\right)_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{-4}{n\pi}\left[\frac{-1}{n}(\pi\cos n\pi - 0)\right] - \frac{4}{n^2\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{+4}{n^2}(-1)^n - \frac{4}{n^3\pi}(\sin n\pi - 0) = \frac{+4}{n^2}(-1)^n - 0$$

$$\therefore \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} x\sin nx\,dx}_{\text{even}} + \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} x^2\sin nx\,dx}_{\text{odd}}$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx + 0$$

$$= \frac{2}{\pi}\left[\left(x\left(\frac{-\cos nx}{n}\right)\right)_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}(\pi\cos n\pi - 0)\right] + \frac{2}{n\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{n^2\pi}(\sin n\pi - 0) = -\frac{2}{n}(-1)^n + 0$$

$$= (-1)\frac{2(-1)^n}{n} = \frac{2}{n}(-1)^{n+1}$$

$$\therefore \boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

$$① \Rightarrow f(x) = \frac{\left(\frac{2\pi^2}{3}\right)}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right] \quad -②$$

__Deduction:__ At $x=0$, $f(x)$ is continuous. Hence the fourier series converges to $f(0)$.

put $x=0$ in ②,
$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\cancel{-}\frac{\pi^2}{3} = \cancel{-}4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

3. Find a fourier series to represent a function, $f(x) = e^x$ in $-\pi < x < \pi$ & hence deduce the series $2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots\right] = \frac{\pi}{\sinh\pi}$

Sol: given $f(x) = e^x$, $-\pi < x < \pi$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2}{\pi}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right)$$

& odd

$$= \frac{2}{\pi}\sinh\pi$$

$$\therefore \boxed{a_0 = \frac{2}{\pi}\sinh\pi} \qquad \left(\because \sinh x = \frac{e^x - e^{-x}}{2}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

WKT $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi} \qquad (a=1,\ b=n)$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{\pi}(\cos n\pi + n\sin n\pi) - e^{-\pi}(\cos n\pi - n\sin n\pi)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right] = \frac{\cos n\pi}{\pi(n^2+1)}(e^{\pi} - e^{-\pi})$$

$$= \frac{2\cos n\pi}{\pi(n^2+1)}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = \frac{2(-1)^n}{\pi(n^2+1)}\sinh\pi$$

$$\boxed{a_n = \frac{2\sinh\pi}{\pi(n^2+1)}(-1)^n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

WKT $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)} \left[e^{\pi} (\sin n\pi - n\cos n\pi) - e^{-\pi} (-\sin n\pi - n\cos n\pi) \right]$$

$$\frac{1}{\pi(n^2+1)} \left[e^{\pi} \sin n\pi - n e^{\pi} \cos n\pi + n e^{-\pi} \cos n\pi \right]$$

$$= \frac{-n\cos n\pi}{\pi(n^2+1)} (e^{\pi} - e^{-\pi}) = \frac{-2n(-1)^n}{\pi(n^2+1)} \left(\frac{e^{\pi} - e^{-\pi}}{2} \right)$$

$(\because \cos n\pi = (-1)^n)$

$$= \frac{-2n(-1)^n}{\pi(n^2+1)} \sinh\pi = \frac{(-1)\,2\sinh\pi}{\pi(n^2+1)} (-1)^n$$

$$\therefore \boxed{b_n = +\frac{2n\sinh\pi}{\pi(n^2+1)} (-1)^{n+1}}$$

① $\Rightarrow$

$$f(x) = \frac{\frac{2}{\pi}\sinh\pi}{2} + \sum_{n=1}^{\infty} \frac{2\sinh\pi}{\pi(n^2+1)} (-1)^n \cos nx + \sum_{n=1}^{\infty} \frac{+2n\sinh\pi}{\pi(n^2+1)} (-1)^{n+1} \sin$$

$$e^x = \frac{1}{\pi}\sinh\pi + \frac{2\sinh\pi}{\pi} \left[-\frac{\cos x}{1^2+1} + \frac{\cos 2x}{2^2+1} - \frac{\cos 3x}{3^2+1} + \cdots \right]$$

$$+ \frac{2\sinh\pi}{\pi} \left[\frac{\sin x}{1^2+1} - \frac{2\sin 2x}{2^2+1} + \frac{3\sin 3x}{3^2+1} \cdots \right]$$

$$e^x = \frac{\sinh\pi}{\pi} \left[1 + 2\left(-\frac{\cos x}{2} + \frac{\cos 2x}{2^2+1} - \frac{\cos 3x}{3^2+1} + \frac{\cos 4x}{4^2+1} - \cdots \right) \right]$$

$$+ \frac{2\sinh\pi}{\pi} \left[\frac{\sin x}{\pi} - \frac{2\sin 2x}{2^2+1} + \frac{3\sin 3x}{3^2+1} - \cdots \right] \quad -②$$

<u>Deduction</u>: At $x=0$, $f(x)$ is continuous. Hence the fourier series converges to $f(0)$

put $x=0$ in ②, we get

$$e^0 = \frac{\sinh\pi}{\pi} \left[1 + 2\left(-\frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \cdots \right) \right] + 0$$

$$1 = \frac{\sinh\pi}{\pi} \left[1 - 1 + 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \cdots \right) \right]$$

$$\Rightarrow 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \cdots \right] = \frac{\pi}{\sinh\pi}$$

20/5/15

4 Find the fourier series of $f(x) = 0, \ -\pi < x < 0$ & hence $= \frac{\pi x}{4}, \ 0 < x < \pi$

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol. given $f(x) = 0, \quad -\pi < x < 0$

$= \frac{\pi x}{4}, \quad 0 \le x < \pi$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -① $$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi x}{4}\right)dx = \frac{1}{4}\left(\frac{x^2}{2}\right)_0^{\pi} = \frac{1}{8}(\pi^2) = \frac{\pi^2}{8}$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{8}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\cos nx\,dx + \int_{0}^{\pi}\frac{\pi x}{4}\cos nx\,dx\right]$$

$$= \frac{1}{4}\int_{0}^{\pi} x\cos nx\,dx = \frac{1}{4}\left[\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{4n}(\pi\sin n\pi - 0) - \frac{1}{4n}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = 0 + \frac{1}{4n^2}(\cos n\pi - 1)$$

$$= \frac{1}{4n^2}((-1)^n - 1)$$

$\therefore a_n = 0$, for n is even

$a_n = \frac{-1}{2n^2}$, for n is odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\,dx + \int_{0}^{\pi}\frac{\pi x}{4}\sin nx\,dx\right]$$

$$= \frac{1}{4}\int_{0}^{\pi} x\sin nx\,dx = \frac{1}{4}\left[\left(x\left(-\frac{\cos nx}{n}\right)\right)_0^{\pi} - \int_{0}^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= -\frac{1}{4n}(\pi\cos n\pi - 0) + \frac{1}{4n}\left(\frac{\sin nx}{n}\right)_0^{\pi} = \frac{-1}{4n}(\pi(-1)^n) + \frac{1}{4n^2}(\sin n\pi - 0)$$

$$= \frac{-\pi}{4n}(-1)^n = \frac{\pi}{4n}(-1)^{n+1}$$

$$\therefore \boxed{b_n = \frac{\pi}{4n}(-1)^{n+1}}$$

$$① \Rightarrow f(x) = \frac{\left(\frac{\pi^2}{8}\right)}{2} + \sum_{n=1,3,5,\dots}^{\infty}\left(\frac{-1}{2n^2}\right)\cos nx + \sum_{n=1}^{\infty}\frac{\pi}{4n}(-1)^{n+1}\sin nx$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] + \frac{\pi}{4}\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right] \quad -②$$

Deduction: At $x=0$, $f(x)$ is continuous

Hence the fourier series of $f(x)$ converges to $f(0)$

Put $x=0$ in ②, we get

$$f(0) = \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]+0 \Rightarrow 0 = \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\Rightarrow \cancel{+}\frac{\pi^2}{\cancel{16}\,8} = \cancel{+}\frac{1}{\cancel{2}}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

Scanned with CamScanner
Scanned with CamScanner

5). If $f(x) = x$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f(x) = 0$ in $(\frac{\pi}{2}, \frac{3\pi}{2})$ find the Fourier series of $f(x)$. Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol:- The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—} ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\,dx + \int_{\pi/2}^{3\pi/2} 0\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left[\cancel{\frac{\pi^2}{4}} - \cancel{\frac{\pi^2}{4}}\right] = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{3\pi/2} 0\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} \quad \left[\because \int uv\,dx = uv_1 - u'v_2 + \cdots\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi/2 \sin n\pi/2}{n}} + \cancel{\frac{\cos n\pi/2}{n^2}} - \left(\cancel{\frac{-\frac{\pi}{2}\sin n(-\pi/2)}{n}} + \cancel{\frac{\cos n(-\pi/2)}{n^2}}\right)\right]$$

$$a_n = 0$$

$$\& \; b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{3\pi/2} 0(\sin nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{(-\cos nx)}{n} - (1)\frac{-\sin nx}{n^2}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[-\frac{\pi/2\cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} - \left(\frac{\pi/2\cos n\pi/2}{n} - \frac{\sin n\pi/2}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi/2 + \frac{2}{n^2}\sin n\pi/2\right]$$

$$\boxed{b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin n\pi/2.}$$

$$\therefore f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left(-\frac{\cos n\pi/2}{n} + \frac{2}{n^2\pi}\sin n\pi/2\right)\sin nx.$$

$$\text{ie. } f(x) = \left[-\frac{\cos\pi/2}{1} + \frac{2}{1^2\pi}\sin\pi/2\right]\sin x + \left[-\frac{\cos\pi}{2} + \frac{2}{2^2\pi}\sin\pi\right]\sin 2x$$
$$+ \left[-\frac{\cos 3\pi/2}{3} + \frac{2}{3^2\pi}\sin 3\pi/2\right]\sin 3x + \cdots$$

$$= \left[-0 + \frac{2}{\pi 1^2}(1)\right]\sin x + \left[-\left(-\frac{1}{2}\right) + 0\right]\sin 2x +$$
$$\left[0 + \frac{2}{3^2\pi}(-1)\right]\sin 3x + \cdots$$

$\therefore f(x) = \frac{2}{\pi}\frac{1}{1^2}\sin x + \frac{1}{2}\sin 2x - \frac{2}{\pi 3^2}\sin 3x + \text{———} \quad (2)$

At $x = \frac{\pi}{2}$ which is a point of discontinuity $\left[\begin{array}{l} \lim_{x\to\frac{\pi}{2}^-} f(x) = \frac{\pi}{2} \\ \lim_{x\to\frac{\pi}{2}^+} f(x) = 0 \end{array}\right]$

$$f\left(\tfrac{\pi}{2}\right) = \frac{1}{2}\left[f\left(\tfrac{\pi}{2}^-\right) + f\left(\tfrac{\pi}{2}^+\right)\right]$$

$\left[\text{L.H.L} \neq \text{R.H.L}\right]$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}$$

~~putting~~ put $x = \frac{\pi}{2}$ in (2), we get

$$f\left(\tfrac{\pi}{2}\right) = \frac{\pi}{4} = \frac{2}{\pi}\frac{1}{1^2}\sin \pi/2 + \frac{1}{2}\sin\pi - \frac{2}{\pi 3^2}\sin\frac{3\pi}{2} + \text{———}$$

i.e. $\frac{\pi}{4} = \frac{2}{\pi}\frac{1}{1^2}(1) + 0 - \frac{2}{\pi 3^2}(-1) + \text{———}$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \text{———}\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \text{———}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$$

ie $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

6) Find the F.S. for $f(x) = -\pi$, for $-\pi < x < 0$
$= x$, for $0 < x < \pi$ &

hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{—} = \frac{\pi^2}{8}$.

7) obtain the F.S. expansion of $f(x) = \frac{\pi - x}{2}$ in $(0, 2\pi)$ &

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{———}$

8) Find the F.S. of $f(x) = 0$, when $-\pi \le x \le 0$
$= x^2$, when $0 \le x \le \pi$

Change of interval

Fourier Series for 2l periodic functions; where l is a +ve constant ~~arbitrary~~ :

Let the function $f(x)$ be periodic of period '2l'. The fourier series of $f(x)$ in the interval $[c, c+2l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$ where

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

21/5/15

1 Find the fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.

Sol. Given $f(x) = 2x - x^2$, $0 < x < 3$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

Here $c = 0$, $c + 2l = 3 \Rightarrow 2l = 3 \Rightarrow \boxed{l = \frac{3}{2}}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2}{3}n\pi\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2}{3}n\pi\right)x \quad \text{---} (1)$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\,dx = \frac{2}{3}\left[2\left(\frac{x^2}{2}\right)_0^3 - \left(\frac{x^3}{3}\right)_0^3\right] = \frac{2}{3}\left[9-\frac{27}{3}\right]$$

$$= \frac{2}{3}(9-9) = 0 \Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left[\left((2x-x^2)\frac{\sin\frac{2n\pi}{3}x}{\left(\frac{2n\pi}{3}\right)}\right)_0^3 - \int_0^3 (2-2x)\frac{\sin\left(\frac{2n\pi}{3}\right)x}{\frac{2n\pi}{3}}\,dx\right]$$

$$= \frac{2}{3}\times\frac{3}{2n\pi}\left[(-3)\sin 2n\pi - 0\right] - \frac{2}{3}\times\frac{3}{2n\pi}\times 2\int_0^3 (1-x)\sin\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= 0 - \frac{2}{n\pi}\left[\left((1-x)\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)}\right)\right)_0^3 - \int_0^3 (-1)\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)}\,dx\right]$$

$$= -\frac{2}{n\pi}\cdot\frac{-3}{2n\pi}\left[(-2)\cos 2n\pi - 1\cdot\cos 0\right] + \frac{2}{n\pi}\cdot\frac{3}{2n\pi}\left(\frac{\sin\left(\frac{2n\pi}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)_0^3$$

$$= \frac{+3}{n^2\pi^2}\left[-2(+1)-1\right] + \frac{3}{n^2\pi^2}\times\frac{3}{2n\pi}\left[\sin 2n\pi - 0\right]$$

$$= \frac{-9}{n^2\pi^2} + 0 \Rightarrow \boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left[\left((2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)}\right)\right)_0^3 - \int_0^3 (2-2x)\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{\frac{2n\pi}{3}}\right)dx\right]$$

$$= \frac{2}{3}\times\frac{-3}{2n\pi}\left[(-3)\cos 2n\pi - 0\right] + \frac{2\times 3}{2n\pi}\times\frac{2}{3}\int_0^3 (1-x)\cos\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= \frac{-1}{n\pi}\left[(-3)1\right] + \frac{2}{n\pi}\left[\left((1-x)\frac{\sin\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)}\right)_0^3 - \int (-1)\frac{\sin\left(\frac{2n\pi}{3}\right)x}{\frac{2n\pi}{3}}\,dx\right]$$

$$= \frac{3}{n\pi} + \frac{2}{n\pi}\cdot\frac{3}{2n\pi}\left[(-2)\sin 2n\pi - 0\right] + \frac{2}{n\pi}\times\frac{3}{2n\pi}\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)}{\frac{2n\pi}{3}}\right)_0^3$$

$$= \frac{3}{n\pi} + \frac{3}{n^2\pi^2}(0) - \frac{3}{n^2\pi^2}\times\frac{3}{2n\pi}\left[\cos 2n\pi - 1\right]$$

Scanned with CamScanner

$$= \frac{3}{n\pi} - \frac{9}{2n^3\pi^3}\left[\pi - \pi\right] = \frac{3}{n\pi} - 0 = \frac{3}{n\pi} \Rightarrow \boxed{b_n = \frac{3}{n\pi}}$$

$$① \Rightarrow f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\left(\frac{2n\pi}{3}\right)x + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\left(\frac{2n\pi}{3}\right)x$$

02/5/15

2. Find the fourier series for $f(x) = 2lx - x^2$ in $(0, 2l)$, hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Sol. given $f(x) = 2lx - x^2$, in $(0, 2l)$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ — ① be the fourier series

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l}(2lx - x^2)\,dx = \frac{1}{l}\left[\frac{2l}{2}(x^2)_0^{2l} - \frac{1}{3}(x^3)_0^{2l}\right]$$

$$= \frac{1}{l}\left[l(4l^2) - \frac{1}{3}(8l^3)\right] = \frac{1}{l}\left[4l^3 - \frac{8l^3}{3}\right] = \frac{1}{l} \times \frac{4l^3}{3} = \frac{4l^2}{3}$$

$$\therefore \boxed{a_0 = \frac{4l^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{2l}(2lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\left((2lx - x^2)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right)_0^{2l} - \int_0^{2l}(2l - 2x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}dx\right]$$

$$= \frac{1}{l}\left[\frac{l}{n\pi}(0 - 0)\right] - \frac{1}{l}\frac{l}{n\pi} \cdot 2\int_0^{2l}(l - x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= -\frac{2}{n\pi}\left[\left((l - x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right)\right)_0^{2l} - \int_0^{2l}(-1)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{-2}{n\pi} \times \frac{-l}{n\pi}\left[(-l)\cos 2n\pi - l\cos 0\right] + \frac{2}{n\pi} \times \frac{l}{n\pi}\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right)_0^{2l}$$

$$= \frac{2l}{n^2\pi^2}\left[-l(1) - l\right] = \frac{4l^2}{n^2\pi} + \frac{2l^2}{n^3\pi^3}\left[\sin 2n\pi - 0\right]$$

$$\left[\because \begin{array}{l}\sin 2n\pi = 0 \\ \cos 2n\pi = 1\end{array}\right]$$

$$= \frac{-4l^2}{n^2\pi^2} \Rightarrow \boxed{a_n = -\frac{4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{2l}(2lx-x^2)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\left[\left((2lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right)_0^{2l} - \int_0^{2l}(2l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{1}{l}\left[\frac{-l}{n\pi}(0-0)\right] + \frac{2}{n\pi}\int_0^{2l}(l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{n\pi}\left[\left((l-x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right)_0^{2l} - \int_0^{2l}(-1)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}dx\right] \quad \left[\because \begin{array}{l}\sin 2n\pi = 0\\ \cos 2n\pi = 1\end{array}\right]$$

$$= \frac{2}{n\pi}\left[\frac{l}{n\pi}\left(-l\sin 2n\pi - l\sin 0\right)\right] + \frac{2l}{n^2\pi^2}\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{2l}$$

$$= \frac{2l}{n^2\pi^2}(0-0) - \frac{2l^2}{n^3\pi^3}(\cos 2n\pi - \cos 0) = 0 - 0 = \underline{0}$$

$$\Rightarrow \boxed{b_n = 0}$$

① $\Rightarrow$

$$f(x) = \frac{2l^2}{3} + \sum_{n=1}^{\infty}\frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + 0$$

$$2lx - x^2 = \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{2\pi}{l}x}{2^2} + \frac{\cos\frac{3\pi}{l}x}{3^2} + \cdots\right] \quad —②$$

<u>Deduction</u>: At $x = l$, $f(x)$ is continuous & hence the fourier series of $f(x)$ converges to $f(l)$

put $x = l$ in ②, we get

$$2l^2 - l^2 = \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$l^2 - \frac{2l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right] \Rightarrow \frac{l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

3. Expand $f(x) = \pi x, \; 0 < x < 1$
$\quad\quad = 0, \; 1 < x < 2$ into fourier series.

Sol. given $f(x) = \pi x, \quad 0 < x < 1$
$\quad = 0, \quad 1 < x < 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

Here $c = 0$, $c + 2l = 2 \Rightarrow 0 + 2l = 2 \Rightarrow \boxed{l = 1}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad —①$$

$a_0 = \frac{1}{l}$

Scanned with CamScanner

Scanned with CamScanner

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 0\,dx$$

$$= \frac{\pi}{2}(x^2)_0^1 = \frac{\pi}{2}$$

$$\Rightarrow \boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{l}\int_0^2 f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 f(x)\cos n\pi x\,dx \quad (\because l = 1)$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 0\cos n\pi x\,dx$$

$$= \pi\int_0^1 x\cos n\pi x\,dx + 0 = \pi\left[\left(x\frac{\sin n\pi x}{n\pi}\right)_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= \pi\left[\frac{1}{n\pi}(\sin n\pi - 0)\right] - \frac{1}{n}\left(\frac{-\cos n\pi x}{n\pi}\right)_0^1 \quad [\because \sin n\pi = 0]$$

$$a_n = 0 + \frac{1}{n^2\pi}(\cos n\pi - \cos 0) = \frac{1}{n^2\pi}((-1)^n - 1) \quad [\cos n\pi = (-1)^n]$$

$$\therefore a_n = 0, \text{ if } n \text{ is even}$$

$$= \frac{-2}{n^2\pi}, \text{ if } n \text{ is odd}$$

$$b_n = \frac{1}{l}\int_0^2 f(x)\sin\frac{n\pi}{l}x\,dx = \int_0^2 f(x)\sin n\pi x\,dx \quad (\because l = 1)$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 (0)\sin n\pi x\,dx = \pi\int_0^1 x\sin n\pi x\,dx$$

$$= \pi\left[\left(x\left(-\frac{\cos n\pi x}{n\pi}\right)\right)_0^1 - \int_0^1\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right]$$

$$= \pi\left[(-1/n\pi)(\cos n\pi - 0)\right] + \frac{1}{n}\left(\frac{\sin n\pi x}{n\pi}\right)_0^1$$

$$= -\frac{1}{n}\cancel{\pi^{-2}}[(-1)^n] + \frac{1}{n^2\pi}(\sin n\pi - 0)$$

$$= n^2\pi^2\frac{(-1)}{n}(-1)^n + 0 = n^2\pi^2\frac{(-1)^{n+1}}{n} \Rightarrow \boxed{b_n = n^2\pi^2\frac{(-1)^{n+1}}{n}}$$

$$① \Rightarrow f(x) = \frac{\left(\frac{\pi}{2}\right)}{2} + \sum_{n=1,3,5,\dots}^{\infty}\frac{-2}{n^2\pi}\cos n\pi x + \sum_{n=1}^{\infty} n^2\pi^2\frac{(-1)^{n+1}}{n}\sin n\pi x$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

$$+ \pi^2\left[\frac{\sin\pi x}{1} - \frac{2^2\sin 2\pi x}{2} + \frac{3^2\sin 3\pi x}{3} \cdots\right]$$

Scanned with CamScanner
Scanned with CamScanner

23/5/15

4. Find the fourier series expansion for $f(x) = 2$, if $-2 \le x \le 0$

$= x$, if $0 < x < 2$

Sol). given $f(x) = 2$, if $-2 \le x \le 0$

$= x$, if $0 < x < 2$

Let $f(x) = \frac{a_0}{2} + \left\{ \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \right.$ —①

Here $c = -2$, $c + 2l = 2 \Rightarrow -2 + 2l = 2 \Rightarrow 2l = 4 \Rightarrow \boxed{l = 2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \quad \text{—①}$$

$$a_0 = \frac{1}{l} \int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right] = \frac{1}{2}\left[2(x)_{-2}^{0} + \frac{2}{\frac{1}{2}}(x^2)_0^2\right]$$

$$= \frac{1}{2}\left[2(2) + \frac{1}{2}(4)\right] = \frac{6}{2} = 3 \Rightarrow \boxed{a_0 = 3}$$

$$a_n = \frac{1}{l}\int_{-2}^{2} f(x)\,dx \cos \frac{n\pi x}{l}\,dx = \frac{1}{2}\left[\int_{-2}^{2} f(x) \cos \frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2 \cos \frac{n\pi x}{2}\,dx + \int_{0}^{2} x \cos \frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right)_{-2}^{0} + \left(x\, \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\right)_0^2 - \int_0^2 \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\,dx\right]$$

$$= \frac{2}{n\pi}(0 + \sin 2n\pi) + \frac{1}{n\pi}(2 \sin n\pi - 0) - \frac{1}{n\pi}\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right)_0^2$$

$$= 0 + 0 + \frac{2}{n^2\pi^2}(\cos n\pi - \cos 0) = \frac{2}{n^2\pi^2}((-1)^n - 1)$$

$\therefore a_n = 0$, if n is even

$= \frac{-4}{n^2\pi^2}$, if n is odd

$$b_n = \frac{1}{l}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2 \sin \frac{n\pi x}{2}\,dx + \int_{0}^{2} x \sin \frac{n\pi x}{2}\,dx\right]$$

$$= \cancel{2}\left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\right)_{-2}^{0} + \frac{1}{2}\left[\left(x\left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\right)\right)_0^2 - \int_0^2 \left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\right) dx\right]$$

$= \frac{-2}{n\pi}(\cos 0 - \cos n\pi) - \frac{1}{n\pi}(2\cos n\pi - 0) + \frac{1}{n\pi}\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right)_0^2$

$= -\frac{2}{n\pi}(1-(-1)^n) - \frac{1}{n\pi}(2(-1)^n) + \frac{2}{n^2\pi^2}(\sin n\pi - 0)$

$= \frac{-2+2(-1)^n - 2(-1)^n}{n\pi} + 0 = \frac{-2}{n\pi} \Rightarrow \boxed{b_n = \frac{-2}{n\pi}}$

① $\Rightarrow f(x) = \frac{3}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty}\frac{-2}{n\pi}\sin\frac{n\pi x}{2}$

$= \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right] - \frac{2}{\pi}\left[\frac{\sin\frac{\pi x}{2}}{1} + \frac{\sin \pi x}{2} + \frac{\sin\frac{3\pi x}{2}}{3} + \cdots\right]$

5. ~~Find~~ Expand $f(x) = e^{-x}$ as a fourier series in the interval $(-1, 1)$.

Sol. given $f(x) = e^{-x}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here $c = -1$, $c + 2l = 1 \Rightarrow -1 + 2l = 1 \Rightarrow 2l = 2 \Rightarrow \boxed{l = 1}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ —①

$a_0 = \frac{1}{l}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} e^{-x}\,dx = -(e^{-x})_{-1}^{1} = -(e^{-1} - e) = e - e^{-1}$

$= \frac{1}{2}(e - e^{-1})\,2$

~~$a_0 = 2\sinh 1$~~

$\boxed{a_0 = 2\sinh 1}$

$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{l}\,dx = \int_{-1}^{1} e^{-x}\cos n\pi x\,dx$

WKT $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

$a = -1$

$b = n\pi$

$= \left(\frac{e^{-x}}{n^2\pi^2+1}\left[-\cos n\pi x + n\pi\sin n\pi x\right]\right)_{-1}^{1}$

$= \frac{e^{-1}}{n^2\pi^2+1}\left[-\cos n\pi + n\pi\sin n\pi\right] - \frac{e^{1}}{n^2\pi^2+1}\left[-\cos n\pi - n\pi\sin n\pi\right]$

$= \frac{e^{-1}}{n^2\pi^2+1}\left[(-1)(-1)^n\right] - \frac{e^{1}}{n^2\pi^2+1}\left[(-1)(-1)^n\right]$

$= \frac{(-1)^{n+1}}{n^2\pi^2+1}(e^{-1} - e^{1}) = \frac{2(-1)^n}{n^2\pi^2+1}\frac{(e - e^{-1})}{2}$

$$\boxed{a_n = \frac{2(-1)^n}{n^2\pi^2+1}\sinh 1}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx = \int_{-1}^{1} e^{-x}\sin n\pi x\,dx$$

WKT $\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ $\quad a=-1,\ b=n\pi$

$$= \left(\frac{e^{-x}}{n^2\pi^2+1}\left[-\sin n\pi x - n\pi\cos n\pi x\right]\right)_{-1}^{1} \qquad \left[\begin{array}{l}\sin n\pi = 0\\ \cos n\pi = (-1)^n\end{array}\right]$$

$$= \frac{e^{-1}}{n^2\pi^2+1}\left[-\sin n\pi - n\pi\cos n\pi\right] - \frac{e^{1}}{n^2\pi^2+1}\left[\sin n\pi - n\pi\cos n\pi\right]$$

$$= \frac{e^{-1}}{n^2\pi^2+1}\left[-n\pi(-1)^n\right] - \frac{e^{1}}{n^2\pi^2+1}\left[-n\pi(-1)^n\right]$$

$$= \frac{n\pi(-1)^n}{n^2\pi^2+1}(e-e^{-1}) = \frac{2n\pi(-1)^n}{n^2\pi^2+1}\left(\frac{e-e^{-1}}{2}\right)$$

$$\boxed{b_n = \frac{2n\pi(-1)^n}{n^2\pi^2+1}\sinh 1}$$

$$① \Rightarrow f(x) = \frac{1}{2}\sinh 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2+1}\sinh 1\cos n\pi x + \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n}{n^2\pi^2+1}\sinh 1\sin n\pi x$$

6. Find the fourier series for the function $f(x) = \pi x,\ 0 \le x \le 1$
$= \pi(2-x),\ 1 \le x \le 2$

Sol). given $f(x) = \pi x,\quad 0 \le x \le 1$
$= \pi(2-x),\quad 1 \le x \le 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$

Here $c=0,\ c+2l=2 \Rightarrow 2l=2 \Rightarrow \boxed{l=1}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n\sin n\pi x \quad —①$$

$$a_0 = \frac{1}{l}\int_0^2 f(x)\,dx = \frac{1}{1}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \frac{\pi}{2}(x^2)_0^1 + \pi\int_1^2(2-x)\,dx = \frac{\pi}{2} + \pi\left[2(x)_1^2 - \frac{1}{2}(x^2)_1^2\right]$$

$$= \frac{\pi}{2} + \pi\left[2 - \frac{1}{2}(3)\right] = \frac{\pi}{2} + 2\pi - \frac{3\pi}{2} = \frac{5\pi-3\pi}{2} = \frac{2\pi}{2}$$

$$\therefore \boxed{\cancel{a_0 = \frac{5\pi-3}{2}}} \qquad \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[\left(x\frac{\sin n\pi x}{n\pi}\right)_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}dx\right] + \pi\left[\left((2-x)\frac{\sin n\pi x}{n\pi}\right)_1^2 - \int_1^2 (-1)\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= \pi\left[\frac{1}{n\pi}(\sin n\pi - 0)\right] - \frac{1}{n}\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1 + \frac{\pi}{n\pi}\left[0 - \sin n\pi\right] - \frac{1}{n}\left(-\frac{\cos n\pi x}{n\pi}\right)_1^2$$

$$= 0 + \frac{1}{n^2\pi^2}(\cos n\pi - 1) + 0 + \frac{1}{n^2\pi}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right] + \frac{1}{n^2\pi}\left[1-(-1)^n\right] = \frac{(-1)^n - 1 + 1 - (-1)^n}{n^2\pi} = 0$$

$$\therefore \boxed{a_n = 0} \qquad \left[\begin{array}{l}\sin n\pi = 0\\ \cos n\pi = (-1)^n\\ \cos 2n\pi = 1\end{array}\right]$$

$$b_n = \frac{1}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left(x\left(\frac{-\cos n\pi x}{n\pi}\right)\right)_0^1 - \int_0^1\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right]$$

$$+ \pi\left[\left((2-x)\left(-\frac{\cos n\pi x}{n\pi}\right)\right)_1^2 - \int_1^2 (-1)\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]$$

$$= \pi\left[\frac{-1}{n\pi}(\cos n\pi - 0)\right] + \frac{1}{n}\left(\frac{\sin n\pi x}{n\pi}\right)_0^1 + \pi\left[\frac{-1}{n\pi}(0 - \cos n\pi)\right] - \frac{1}{n}\left(\frac{\sin n\pi x}{n\pi}\right)_1^2$$

$$= -\frac{1}{n}\cancel{(-1)^n} + \frac{1}{n^2\pi}(\sin n\pi - 0) + \frac{1}{n}\cancel{(-1)^n} - \frac{1}{n^2\pi}(\sin 2n\pi - \sin n\pi)$$

$$0 - 0 = 0 \quad \Rightarrow \boxed{b_n = 0}$$

① $\Rightarrow$ $f(x) = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}$

15/6/15

~~Even~~ Fourier Series for Even & Odd Functions

Even Function: A function $f(x)$ is said to be even if $f(-x) = f(x)$ $\forall x$

Ex x^2, $\cos x$ are even functions of x.

Note 1) The graph of an even function is symmetrical about the y-axis. 2). Modulas function is always an even function i.e. $|x|$, $|\sin x|$, $|\cos x|$ etc are even functions.

Odd Function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex. x^3, $\sin x$, x are ~~even~~ odd functions of x

Note. The graph of an odd function is symmetrical about the origin.

Fourier Series for even & odd function:

→ If $f(x)$ is even in $(-\pi, \pi)$ then it's fourier series is in the form of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, $\left[\because b_n = 0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{(even)}{f(x)} \underset{(odd)}{\sin nx}\, dx = odd \right]$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$

→ If $f(x)$ is odd in $(-\pi, \pi)$ then it's fourier series is of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

→ If $f(x)$ is even in $(-l, l)$ then it's fourier series is in the form of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$, $\left[\because a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{(odd)}{f(x)}\, dx = 0,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{(odd)}{f(x)} \underset{(even)}{\cos nx}\, dx = 0 = odd. \right]$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx$, $a_n = \frac{2}{l}\int_0^{l} f(x) \cos \frac{n\pi x}{l}\, dx$.

→ If $f(x)$ is odd in $(-l, l)$ then it's fourier series is of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, where $b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l}\, dx$

Formula

1) If $f(x)$ is defined in $[-a, a]$ then

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^{a} f(x)\, dx, \text{ when } f(x) \text{ is an even function}$$

$$= 0, \text{ when } f(x) \text{ is an odd function}$$

1. Find the fourier series to represent the function $f(x) = x^2 - 2$, $-2 \le x \le 2$

Sol given $f(x) = x^2 - 2$

$f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$

$\therefore f(x)$ is an even function & it's Fourier series contains a_0 & a_n only.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

Here $l = 2$, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$ —①

where $a_0 = \frac{2}{l} \int_0^l f(x)\,dx = \frac{2}{2} \int_0^2 (x^2 - 2)\,dx \quad (\because l = 2)$

$= \frac{1}{3}(x^3)_0^2 - 2(x)_0^2 = \frac{8}{3} - 4 = \frac{-4}{3} \Rightarrow \boxed{a_0 = -4/3}$

$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,dx = \frac{2}{2} \int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\,dx$

$= \left[(x^2-2) \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)} - 2x \left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) + 2 \left(\frac{-\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3} \right) \right]_0^2$

$= \left[2 \sin n\pi \left(\frac{2}{n\pi}\right) + 4\left(\frac{4}{n^2\pi^2}\right) \cos n\pi - 2\left(\frac{8}{n^3\pi^3}\right) \sin n\pi \right] - [0 + 0 - 0]$

$= \frac{4}{n\pi}(0) + \frac{16}{n^2\pi^2}(-1)^n - \frac{16}{n^3\pi^3}(0) \quad \left[\begin{array}{l} \because \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array} \right]$

$\boxed{a_n = \frac{16}{n^2\pi^2}(-1)^n}$

[$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$
where ' denotes diff & suffix denotes integration]

① $\Rightarrow f(x) = \frac{\left(-\frac{4}{3}\right)}{2} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n \cos \frac{n\pi x}{2}$

$= -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n \cos \frac{n\pi x}{2}$

16/6/15

2. Obtain the fourier series for $f(x) = |x|$ in $[-\pi, \pi]$, hence

Show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

Sol. given $f(x) = |x|$

$$f(-x) = |-x| = x = f(x)$$

$\therefore f(x)$ is an even function & it's F.S contains a_0 & a_n only.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — ①

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$ $\left(\because |x| = -x, x<0 \atop = +x, x\geq 0\right)$

$$= \frac{2}{\pi}\left(\frac{1}{2}\right)(x^2)_0^{\pi} = \frac{1}{\pi}(\pi^2) = \pi \Rightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{\cancel{\pi}}\sin n\pi + \frac{\cos n\pi}{n^2} - \left(0+\frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[0+\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right] \Rightarrow a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

① $\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right]$

$\therefore a_n = 0$, if n is even

$= \frac{-4}{n^2\pi}$, if n is odd

① $\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\frac{-4}{n^2\pi}\cos nx$

$$|x| = \frac{\pi}{2} + \frac{-4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right] \text{ — ②}$$

Deduction: $f(x)$ is continuous at $x=0$ & hence fourier series converges to $f(0)$.

② $\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right] \Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

3. Expand $f(x) = 1 + \frac{2x}{\pi}$ if $-\pi \le x \le 0$ as a fourier series,

$= 1 - \frac{2x}{\pi}$ if $0 \le x \le \pi$

hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

sol.

$$f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} \cancel{= f(x)} \text{ in } [-\pi, 0]$$

$$= f(x) \text{ in } [0, \pi]$$

$$f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} \text{ in } [0, \pi]$$

$$= f(x) \text{ in } [-\pi, 0]$$

$\therefore$ $f(x)$ is an even function in $[-\pi, \pi]$ & it's F.S contains a_0 & a_n only.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx$ — ①

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[(x)_0^{\pi} - \frac{2}{2\pi}(x^2)_0^{\pi}\right] = \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi}\right] = 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\sin n\pi}{n} - \frac{2}{n^2\pi}\cos n\pi + \frac{2}{n^2\pi}\right]$$

$$= \frac{2}{\pi}\left[-0 - \frac{2}{n^2\pi}(-1)^n + \frac{2}{n^2\pi}\right] = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$\therefore$ $a_n = 0$, if n is even

$= \frac{8}{n^2\pi^2}$, if n is odd

$$① \Rightarrow f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8}{n^2\pi^2}\cos nx = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right.$$

$= 1 - \frac{2x}{\pi} - \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$ — ②

Deduction $f(x)$ is continuous at $x=0$ as hence it's F.S is converged to $f(0)$.

put $x=0$ in ②

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

17/6/15

4. Obtain Fourier series for the function $f(x)$ given by

$f(x) = -\frac{1}{2}(\pi + x)$ for $-\pi < x < 0$

$= \frac{1}{2}(\pi - x)$ for $0 < x < \pi$

Sol. $f(-x) = -\frac{1}{2}(\pi - x)$ in $(-\pi, 0)$

$= -f(x)$ in $(0, \pi)$ ie. $f(-x) = -f(x)$ in $(-\pi, \pi)$

$f(-x) = \frac{1}{2}(\pi + x)$ in $(0, \pi)$

$= -f(x)$ in $(-\pi, 0)$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$ & it's F.S contains b_n only.

let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ — ①

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\pi - x) \sin nx \, dx$

$= \frac{1}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx = \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{1}{\pi}\left[0 - \underset{\downarrow\,0}{\frac{\sin n\pi}{n^2}} + \frac{\pi}{n}\cos 0 + \sin 0\right] = \frac{1}{\pi}\left[\frac{\pi}{n}\right] = \frac{1}{n}$

$\Rightarrow \boxed{b_n = \frac{1}{n}}$

① $\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$

5

① Expand the function $f(x) = x^2$ as a fourier series in $[-\pi, \pi]$ and hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

Sol: Given function $f(x) = x^2$ defined in $[-\pi, \pi]$

Clearly $f(x)$ is an even function in $[-\pi, \pi]$

W.K.T Fourier series expansion of an even function in $[-\pi, \pi]$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — ①

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi} \int_0^{\pi} x^2\,dx = \frac{2}{\pi} \left(\frac{x^3}{3}\right)_0^{\pi}$

$\Rightarrow a_0 = \frac{2}{\pi} \left(\frac{\pi^3}{3} - 0\right) = \frac{2\pi^2}{3}$.

Scanned with CamScanner

Scanned with CamScanner

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\,\underset{v}{\cos nx}\,dx$$

$$= \frac{2}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]$$

Here $u = x^2$ $\quad v = \cos nx$

$v_1 = \int v\,dx = \sin nx/n$

$u' = 2x$ $\quad v_2 = \int v_1\,dx = -\cos nx/n^2$

$u'' = 2$ $\quad v_3 = \int v_2\,dx = -\sin nx/n^3$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right] \qquad (\because \sin n\pi = 0,\ \cos n\pi = (-1)^n,\ \sin 0 = 0).$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Sub a_0, a_n values in ①, we have

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx. \quad \text{②}$$

Deductions (i) put $x=0$ in ② we have

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Scanned with CamScanner
Scanned with CamScanner

$$\Rightarrow \frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots=\frac{-\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$$

(ii) put $x=\pi$ in (2) we have.

$$\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}=\pi^2-\frac{\pi^2}{3}$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}=\frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \qquad (\because (-1)^{2n}=1)$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

(iii) adding (i) & (ii) we have

$$\left(\frac{1}{1^2}-\cancel{\frac{1}{2^2}}+\frac{1}{3^2}-\cancel{\frac{1}{4^2}}+\cdots\right)+\left(\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cancel{\frac{1}{4^2}}+\cdots\right)=\frac{\pi^2}{12}+\frac{\pi^2}{6}$$

$$\Rightarrow 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=\frac{\pi^2+2\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{3\pi^2}{2\times 12}$$

$$\Rightarrow \boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}}$$

18/6/15

Half-Range Series: In the previous section, we have considered the fourier series expansion of a function $f(x)$ defined in $(-l,l)$ of length $2l$ (or) $(-\pi,\pi)$ of length 2π. In various engineering problems, it is required to obtain a fourier series of $f(x)$ defined in half of the above interval say $(0,l)$ (or) $(0,\pi)$. Such expansions (or) series is known as Half-Range Series.

As it is immaterial, whatever the function may be outside the range, $0<x<l$. We extend the function to cover the range $-l<x<l$. So, the new function becomes even (or) odd. Therefore the fourier series of such functions having cosine terms (or) sine terms only.

The expansion containing only cosine terms is known as "Half-range cosine series."

The expansion containing only sine terms is known as "Half-range sine series."

Half-range Cosine Series: The half-range cosine series of $f(x)$ in the interval $(0,l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

Half-range Sine Series: The half-range sine series of $f(x)$ in the interval $(0,l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Scanned with CamScanner
Scanned with CamScanner

Note: It is required to find the half-range fourier cosine (or) sine series of $f(x)$ in $(0, \pi)$ by replacing 'l' with 'π' in the above formulae

1. Find the half-range fourier sine series for $f(x) = x(\pi - x)$ in $(0, \pi)$ & hence deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}$

Sol: The half-range sine series of $f(x)$ in the interval $(0, \pi)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ — ①

where $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^\pi (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) \right]_0^\pi + (-2)\left(\frac{\cos nx}{n^3}\right)\Bigg]_0^\pi$$

$$= \frac{2}{\pi}\left[0 + (-\pi)\frac{\sin n\pi}{n^2} - \frac{2}{n^3}\cos n\pi + 0 - \pi(0) + \frac{2}{n^3} \right]$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n^3}(-1)^n + \frac{2}{n^3} \right] = \frac{4}{n^3 \pi}\left[1 - (-1)^n\right]$$

$b_n = 0$, if n is even

$= \frac{8}{n^3 \pi}$, if n is odd

① $\Rightarrow f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8}{n^3 \pi} \sin nx = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots \right]$

$\therefore x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots \right]$ — ②

Deduction: $f(x)$ is continuous at $x = \pi/2$ & hence its F.S is converged to $f(\pi/2)$. put $x = \pi/2$ in ②, we get

② $\Rightarrow \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \right]$

$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \right]$

$\therefore \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^3}{32}$

19/6/19

2. Find the half-range sine series for $f(x) = 1, \quad 0 < x < \pi/2$

$= -1, \quad \pi/2 < x < \pi$

Sol) The half-range fourier sine series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad —①$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\sin nx\, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \sin nx\, dx - \int_{\pi/2}^{\pi} \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + \left(\frac{\cos nx}{n}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi/2}{n} + \frac{1}{n} + \frac{\cos n\pi}{n} - \frac{\cos n\pi/2}{n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} + \frac{1}{n}(-1)^n - \frac{2}{n}\cos n\pi/2\right]$$

$$b_n = \frac{2}{n\pi}\left[1 + (-1)^n - 2\cos n\pi/2\right]$$

$$① \Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 + (-1)^n - 2\cos n\pi/2\right] \sin nx$$

3. Find the half-range fourier sine series of $f(x) = e^x$ in $(0,1)$.

Sol) The half range fourier sine series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad —① \qquad \text{Here } l = 1$$

where $b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l} \, dx$

Scanned with CamScanner
Scanned with CamScanner

$$= \frac{2}{1}\int_0^1 e^x \sin n\pi x \, dx = 2\left[s \right.$$

WKT $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$$

$$= 2\int_0^1 \left[\frac{e^x}{n^2\pi^2+1}(\sin n\pi x - n\pi \cos n\pi x)\right]_0^1 \qquad \begin{matrix} a=1 \\ b=n\pi \end{matrix}$$

$$= 2\left[\frac{e}{n^2\pi^2+1}(\sin n\pi - n\pi \cos n\pi) - \frac{1}{n^2\pi^2+1}(0 - n\pi)\right]$$

$$b_n = 2\left[\frac{e}{n^2\pi^2+1}(-n\pi(-1)^n) + \frac{n\pi}{n^2\pi^2+1}\right]$$

$$b_n = \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right]$$

$$① \Rightarrow \quad f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1+n^2\pi^2}(1-e(-1)^n) \sin n\pi x.$$

4. Find the half-range fourier cosine series of $f(x)=x$ in $0<x<\pi$.

Sol. The half-range fourier cosine series for $f(x)$ in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad — ①$$

Where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}(\pi^2 - 0) = \pi \Rightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}((-1)^n - 1)\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$\therefore \; a_n = 0$, if n is even

$= \frac{-4}{n^2\pi}$, if n is odd

Scanned with CamScanner

Scanned with CamScanner

Scanned with CamScanner

$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2\pi} \cos nx = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\cos nx}{n^2}$

5. Obtain fourier cosine series for $f(x) = x \sin x$, $0 < x < \pi$

Sol: The half-range fourier cosine series for $f(x)$ in $(0,\pi)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — ①

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \,dx$

$= \frac{2}{\pi} \left[x(-\cos x) + \sin x \right]_0^{\pi} = \frac{2}{\pi} \left[-\pi \cos\pi + \sin\pi + 0 \right]$

$= \frac{2}{\pi} [\pi] = 2 \Rightarrow \boxed{a_0 = 2}$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \,dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \,dx$

$= \frac{1}{\pi} \int_0^{\pi} x \,(2 \cos nx \sin x)\, dx$

$= \frac{1}{\pi} \int_0^{\pi} x \left[\sin(nx + x) - \sin(nx - x) \right] dx$

$= \frac{1}{\pi} \int_0^{\pi} x \sin(n+1)x \,dx - \frac{1}{\pi} \int_0^{\pi} x \sin(n-1)x \,dx$

$= \frac{1}{\pi} \left[x \left(\frac{-\cos(n+1)x}{n+1} \right) + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{\pi} - \frac{1}{\pi} \left[x \left(\frac{-\cos(n-1)x}{n-1} \right) + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{\pi}$

$= \frac{1}{\pi} \left[\frac{-\pi}{n+1} \cos(n+1)\pi + \frac{\sin(n+1)\pi}{(n+1)^2} - (0+0) \right] - \frac{1}{\pi} \left[\frac{-\pi}{n-1} \cos(n-1)\pi + \frac{\sin(n-1)\pi}{(n-1)^2} - (0+0) \right]$

$= \frac{-1}{n+1} \cos(n+1)\pi + \frac{\sin(n+1)\pi}{\pi(n+1)^2} + \frac{1}{n-1} \cos(n-1)\pi - \frac{\sin(n-1)\pi}{\pi(n-1)^2}$

$= \frac{-1}{n+1} (-1)^{n+1} + \frac{1}{n-1} (-1)^{n-1} = \frac{+1}{n+1} (-1)^n + \frac{1}{n-1} \frac{(-1)^n}{n-1}$

$= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1} \right] = (-1)^n \left(\frac{n-1-n-1}{n^2-1} \right) = \frac{2(-1)^{n+1}}{n^2-1}, \; n \neq$

Scanned with CamScanner
Scanned with CamScanner

if $n=1$, then the above integral becomes

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \; 2\sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{x}{2}(-\cos 2x) + \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\cos 2\pi + \frac{\sin 2\pi}{4} \right] = -\frac{1}{2}(1) = -\frac{1}{2} \Rightarrow \boxed{a_1 = -\frac{1}{2}}$$

① $\Rightarrow$ $f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}(-1)^{n+1} \cos nx$

Scanned with CamScanner

Scanned with CamScanner

1. Expand the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in Fourier series in the interval $(-\pi, \pi)$.

2. Find the F.S. of $f(x) = x^3$ in $(-\pi, \pi)$.

3. Find the F.S. of $f(x) = -K$, if $-\pi < x < 0$
$= K$, if $0 < x < \pi$.

4. Find the F.S. of $f(x) = |\sin x|$ in $(-\pi, \pi)$

5. Prove that the interval $(-\pi, \pi)$,
$$x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1} \sin nx.$$

6. Obtain the F.S. for $\sqrt{1-\cos x}$ in the interval $-\pi \le x \le \pi$.

7. Find the Half range cosine series for the function $f(x) = (x-1)^2,\ 0 \le x \le 1$. Hence deduce that
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

8. Find the Half range sine series of the function
$$f(x) = \frac{1}{4} - x,\quad 0 < x < \frac{1}{2}$$
$$= x - \frac{3}{4},\quad \frac{1}{2} < x < 1$$

Scanned with CamScanner

Scanned with CamScanner

Fourier Transforms

Fourier integral: A periodic function $f(x)$ defined in a finite interval $(-l, l)$ can be expressed in Fourier series. By extending this concept, non-periodic functions defined in $-\infty < x < \infty$ ($\forall x$) can be expressed as a Fourier integral

Fourier integral theorem:

St: If $f(x)$ satisfies Dirichlet's conditions in every interval $(-l, l)$ and $f(x)$ is absolutely integrable in the interval $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi} \int_{\alpha=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos\alpha(t-x)\, dt\, d\alpha \qquad \left[\begin{array}{c}\text{we will take} \\ \alpha = \lambda = p\end{array}\right]$$

(or) $$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos\alpha(t-x)\, dt\, d\alpha \quad - \text{I}$$

$\left[\because \cos\alpha(t-x) \text{ is an even function of } \alpha \text{ then } \int_{-\infty}^{\infty} \cos\alpha(t-x)\, d\alpha = 2\int_0^{\infty} \cos\alpha(t-x)\, d\alpha\right]$

which is valid at all points of continuity

Fourier sine and cosine integrals

w.k.t. $\cos\alpha(t-x) = \cos\alpha t \cos\alpha x + \sin\alpha t \sin\alpha x$ $\quad \left[\cos(A-B) = \cos A \cos B + \sin A \sin B\right]$

$\therefore$ Fourier integral of $f(x)$ can be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \left[\cos\alpha t \cos\alpha x + \sin\alpha t \sin\alpha x\right] dt\, d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos\alpha x \left[\int_{-\infty}^{\infty} f(t) \cos\alpha t\, dt\right] d\alpha + \frac{1}{\pi} \int_0^{\infty} \sin\alpha x \left[\int_{-\infty}^{\infty} f(t) \sin\alpha t\, dt\right] d\alpha \quad - (1)$$

Case(i):- when $f(t)$ is an odd function, $f(t)\cos\alpha t$ is odd while $f(t)\sin\alpha t$ is even.

The 1st integral in (1) is zero and, we get $\quad \left[\because \int_{-a}^{a} f(x)\, dx = 0 \text{ when } f(x) \text{ is odd}\right]$

$$\boxed{\therefore f(x) = \frac{2}{\pi} \int_0^{\infty} \sin\alpha x \left[\int_0^{\infty} f(t) \sin\alpha t\, dt\right] d\alpha}$$

This is called Fourier sine integral.

Case(ii): when $f(t)$ is an even function, $f(t)\cos\alpha t$ is even, while $f(t)\sin\alpha t$ is odd

The second integral in (1) is zero and, we get

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \cos\alpha x \left[\int_0^{\infty} f(t) \cos\alpha t\, dt\right] d\alpha.}$$

This is called Fourier cosine integral.

Note for I:- At a point of discontinuity x_0, the Fourier integral $= \frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right]$ i.e. average of the left and right hand limits.

1). Express the function $f(x) = 1$ for $|x| \le 1$
$= 0$ for $|x| > 1$ as a Fourier integral

and hence evaluate $\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha$.

Sol - The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos\alpha(t-x)\, dt\, d\alpha$$

Let us take $f(t) = 1$, for $|t| \le 1$
$= 0$ for $|t| > 1$

$\left[\because |t| \le 1,\ -1 \le t \le 1 \atop |t|>1,\ t>1 \text{ or } -t>1 \atop \text{i.e. } t>1,\ t<-1 \right]$

$-\infty \quad -1 \quad 1 \quad \infty$

$$\therefore f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{-1} 0\, dt + \int_{-1}^{1} (1) \cos\alpha(t-x)\, dt + \int_1^\infty (0)\, dt \right] d\alpha$$

$\left[\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} \right]$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{\sin\alpha(t-x)}{\alpha} \right]_{-1}^{1} d\alpha$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\alpha} \left[\sin\alpha(1-x) - \sin\alpha(-1-x) \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\alpha} \left[\sin(\alpha - \alpha x) + \sin(\alpha + \alpha x) \right] d\alpha$$

$\left[\sin(A+B) + \sin(A-B) = 2\sin A \cos B \right]$

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{\alpha}\, 2\sin\alpha \cos\alpha x\, d\alpha.$$ is the required Fourier integral $\quad \text{—} (1)$

To find $\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha$: From (1), we have,

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{2\sin\alpha \cos\alpha x}{\alpha} d\alpha$$

i.e. $$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha$$

$$\therefore \int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha = \frac{\pi}{2} f(x) = \frac{\pi}{2},\ |x| \le 1$$
$$= 0,\ |x| > 1.$$

2) Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos x\alpha + \alpha \sin x\alpha}{1+\alpha^2} d\alpha = \begin{cases} 0, & \text{if } x<0 \\ \frac{\pi}{2}, & \text{if } x=0 \\ \pi e^{-x}, & \text{if } x>0 \end{cases}$$

Sol:- The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right]d\alpha$$

Let us take $f(x) = 0$, if $x<0$ (or) $f(t) = 0$, if $t<0$

$= \pi e^{-x}$, if $x>0$ $\quad = \pi e^{-t}$ if $t>0$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{0} 0\cdot\cos\alpha(t-x)\,dt + \int_0^\infty \pi e^{-t}\cos\alpha(t-x)\,dt\right]d\alpha$$

$$= \int_0^\infty \left[\int_0^\infty e^{-t}\cos\alpha(t-x)\,dt\right]d\alpha \quad \left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\right]$$

$$= \int_0^\infty \left[\frac{e^{-t}}{(-1)^2+\alpha^2}\left[-\cos\alpha(t-x) + \alpha\sin\alpha(t-x)\right]\right]_0^\infty d\alpha$$

$$= \int_0^\infty \left[0 - \frac{e^{-0}}{1+\alpha^2}\left(-\cos\alpha x + \alpha\sin\alpha(0-x)\right)\right]d\alpha$$

$$= \int_0^\infty \left[-\frac{1}{1+\alpha^2}\left(-\cos\alpha x - \alpha\sin\alpha x\right)\right]d\alpha \quad \left[\sin(-\theta) = -\sin\theta\right]$$

$$f(x) = \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha$$

~~At x=0~~ $\Rightarrow \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = 0$ if $x<0$

$= \pi e^{-x}$, if $x>0$

At $x=0$, $f(x)$ has a discontinuity. So $f(x) = \frac{1}{2}\left[f(0^+) + f(0^-)\right]$

i.e. $f(x) = \frac{1}{2}\left[\pi + 0\right] = \frac{\pi}{2}$

$$\therefore \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = \frac{\pi}{2}, \text{ if } x=0.$$

$$\therefore \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = \begin{cases} 0, & \text{if } x<0 \\ \frac{\pi}{2}, & x=0 \\ \pi e^{-x}, & \text{if } x>0. \end{cases}$$

Find the Fourier integral representation for the following functions

i) $f(x) = \frac{\pi}{2}\cos x, \quad |x| \le \pi$

$\quad = 0 \quad |x| > \pi$

ii) $f(x) = 1, \quad |x| \le a$

$\quad = 0, \quad |x| > a.$

iii) $f(x) = e^{-|x|}, \quad -\infty < x < \infty$

Scanned with CamScanner

Scanned with CamScanner

8/7/14

2. Using fourier cosine integral formula, prove that

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos \alpha x}{a^2+\alpha^2}\, d\alpha, \quad a>0, x \geq 0$$

Sol:- The fourier cosine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x \left[\int_0^\infty f(t)\cos\alpha t\, dt\right] d\alpha$$

let us take $f(t) = e^{-at}$ then

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x \left[\int_0^\infty e^{-at}\cos\alpha t\, dt\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \cos\alpha x \left[\frac{e^{-at}}{a^2+\alpha^2}\left(-a\cos\alpha t + \alpha\sin\alpha t\right)_0^\infty\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \cos\alpha x \left[0 - \frac{1}{a^2+\alpha^2}\left(-a(1) + \alpha(0)\right)\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{a\cos\alpha x}{a^2+\alpha^2}\, d\alpha$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\alpha x}{a^2+\alpha^2}\, d\alpha$$

3. Using fourier integral formula, prove that

$$\int_0^\infty \frac{(\alpha^2+2)\cos\alpha x}{\alpha^4+4}\, d\alpha = \frac{\pi}{2} e^{-x}\cos x.$$

Sol: Since the integrant on LHS contains cosine terms, we use fourier cosine integral formula.

$\therefore$ By fourier cosine integral formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x \left[\int_0^\infty f(t)\cos\alpha t\, dt\right] d\alpha$$

let us take $f(t) = e^{-t}\cos t$ then

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\left[\int_0^\infty e^{-t}\cos t \cos\alpha t\, dt\right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\int_0^\infty e^{-t}(2\cos\alpha t\cos t)\, dt\right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\int_0^\infty e^{-t}\left[\cos(\alpha+1)t + \cos(\alpha-1)t\right] dt\right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\int_0^\infty e^{-t}\cos(\alpha+1)t\, dt + \int_0^\infty e^{-t}\cos(\alpha-1)t\, dt\right] d\alpha \quad \left[\because \int_0^\infty e^{-ax}\cos bx = \frac{a}{a^2+b^2}\right]$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\frac{1}{1^2+(\alpha+1)^2} + \frac{1}{1^2+(\alpha-1)^2}\right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\frac{1}{\alpha^2+2\alpha+2} + \frac{1}{\alpha^2-2\alpha+2}\right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \cos\alpha x\left[\frac{2\alpha^2+4}{(\alpha^2+2)^2-4\alpha^2}\right] d\alpha$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,\frac{(\alpha^2+2)}{\alpha^4+4}\, d\alpha$$

$$\therefore e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,\frac{(\alpha^2+2)}{\alpha^4+4}\, d\alpha$$

$$\therefore \int_0^\infty \cos\alpha x\,\frac{(\alpha^2+2)}{(\alpha^4+4)}\, d\alpha = \frac{\pi}{2} e^{-x}\cos x.$$

4. Using fourier sine integral formula, prove that

$$\int_0^\infty \frac{\alpha\sin\alpha x}{a^2+\alpha^2}\, d\alpha = \frac{\pi}{2} e^{-ax},\quad a>0,\ x>0.$$

Sol: The fourier sine integral formula of $f(x)$ is

Scanned with CamScanner

Scanned with CamScanner

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\int_0^\infty f(t)\sin\alpha t\, dt\right] d\alpha$$

let us consider $f(t) = e^{-at}$ then

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\int_0^\infty e^{-at}\sin\alpha t\, dt\right] d\alpha$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \sin\alpha x \quad \frac{\alpha}{a^2+\alpha^2}\, d\alpha \qquad \left[\because \int_0^\infty e^{-at}\sin bt\, dt = \frac{b}{a^2+b^2}\right.$$

$$e^{-ax}\cdot\frac{\pi}{2} = \int_0^\infty \frac{\alpha\sin\alpha x}{a^2+\alpha^2}\, d\alpha$$

5. Using fourier sine integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\alpha\sin\alpha x}{(\alpha^2+a^2)(\alpha^2+b^2)}\, d\alpha \qquad a, b > 0$$

Sol: The fourier sine integral formula for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\int_0^\infty f(t)\, \sin\alpha t\, dt\right] d\alpha$$

let $f(t) = e^{-at} - e^{-bt}$ then

$$= \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\int_0^\infty (e^{-at}-e^{-bt})\sin\alpha t\, dt\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\int_0^\infty e^{-at}\sin\alpha t\, dt - \int_0^\infty e^{-bt}\sin\alpha t\, dt\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \sin\alpha x \left[\frac{\alpha}{a^2+\alpha^2} - \frac{\alpha}{b^2+\alpha^2}\right] d\alpha$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \alpha\sin\alpha x \left[\frac{b^2+\alpha^2-a^2-\alpha^2}{(a^2+\alpha^2)(b^2+\alpha^2)}\right] d\alpha$$

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}(b^2-a^2)\int_0^\infty \frac{\alpha\sin\alpha x}{(\alpha^2+a^2)(\alpha^2+b^2)}\, d\alpha$$

Scanned with CamScanner

Scanned with CamScanner

1. Express $f(x) = 1$ for $0 \le x \le \pi$
$= 0$ for $x > \pi$
as a Fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha x \, d\alpha$.

2. Using Fourier cosine integral, prove that $\int_0^\infty \frac{\cos\alpha x}{1+\alpha^2} d\alpha = \frac{\pi}{2}e^{-x}$, $x \ge 0$.

Complex (or) Exponential form of fourier integral

The complex form of fourier integral is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha(t-x)}\, dt\, d\alpha$$

Proof:- By fourier integral formula, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\,d\alpha$ — ①

Here $\cos\alpha(t-x)$ is an even function of α.

Since $\sin\alpha(t-x)$ is an odd function of α $\left[\int_{-\infty}^{\infty}\sin\alpha(t-x)\,d\alpha = 0\right]$

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\,d\alpha \quad — ②$$

① + i (2), we get

i.e. $f(x) + i(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right]d\alpha + i\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\right]d\alpha$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\left[\cos\alpha(t-x) + i\sin\alpha(t-x)\right]dt\right]d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha(t-x)}\,dt\,d\alpha \quad \left[\cos\theta + i\sin\theta = e^{i\theta}\right]$$

— ③

which is the required complex form.

③ can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, e^{-i\alpha x}\,dt\,d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\,dt\right] e^{-i\alpha x}\,d\alpha \quad — \text{I}$$

Infinite Fourier Transform:

Denote the value of inner integral (I) by $F(\alpha)$

i.e. $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\,d\alpha$ — ④

where $\boxed{F(\alpha) = \int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\,dt}$ is called Fourier Transform of $f(x)$ and is denoted by $F[f(x)]$

And (4) is called Inverse Fourier transform of $F(\alpha)$

i.e. $F^{-1}[F(\alpha)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha.$

Fourier sine and cosine transforms

Fourier sine transform: By Fourier sine integral formula

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\alpha x \left[\int_0^{\infty} f(t)\sin\alpha t\, dt\right] d\alpha$$

Denote the value of inner integral by $F_s(\alpha)$

i.e. $\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\alpha x\, F_s(\alpha)\, d\alpha}$ — (1)

where $\boxed{F_s(\alpha) = \int_0^{\infty} f(t)\sin\alpha t\, dt}$ is called Fourier sine transform of $f(x)$ and is denoted by $F_s[f(x)]$. and (1) is called Inverse Fourier sine transform of $F_s(\alpha)$. $[\because F_s(\alpha) = F_s[f(x)]]$

Fourier cosine transform: By Fourier cosine integral formula

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\alpha x \left[\int_0^{\infty} f(t)\cos\alpha t\, dt\right] d\alpha$$

Denote the value of inner integral by $F_c(\alpha)$

i.e. $\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\alpha x\, F_c(\alpha)\, d\alpha}$ — (2)

where $\boxed{F_c(\alpha) = \int_0^{\infty} f(t)\cos\alpha t\, dt}$ is called Fourier cosine transform, and (2) is called Inverse Fourier cosine transform of $F_c(\alpha)$. $[F_c(\alpha) = F_c[f(x)]]$

Properties.

1) $F_s[x f(x)] = -\frac{d}{d\alpha}[F_c(\alpha)]$

2) $F_c[x f(x)] = \frac{d}{d\alpha}[F_s(\alpha)]$

3) $F\left[\frac{d^n}{dx^n} f(x)\right] = (-i\alpha)^n F(\alpha)$

4) $F_s[f'(x)] = -\alpha F_c(\alpha)$

5) $F_c[f'(x)] = \alpha F_s(\alpha) - f(0)$

10/7/14

Properties of fourier transforms :-

1. linearity property :-

st :- If $F(\alpha)$ & $G(\alpha)$ be the fourier transforms of $f(x)$ & $g(x)$ respectively then

$$F[af(x) + b\, g(x)] = aF(\alpha) + b\, G(\alpha).$$

Proof: By the definition $F(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx$

$$G(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} g(x)\, dx \quad \text{—(2)}$$

Scanned with CamScanner

Scanned with CamScanner

Scanned with CamScanner

$$\therefore F[af(x)+bg(x)] = \int_{-\infty}^{\infty} e^{i\alpha x}[af(x)+bg(x)]\,dx$$

$$= a\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx + b\int_{-\infty}^{\infty} e^{i\alpha x} g(x)\,dx$$

$$= a\,F(\alpha) + b\,G(\alpha)$$

2. change of Scale property:

St: If $F(\alpha)$ is the fourier transform of $f(x)$ then $F[f(ax)] = \frac{1}{a}F\left(\frac{\alpha}{a}\right)$ where $a>0$

Proof:- By definition

$$F[\alpha] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$$

$$\therefore F[f(ax)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(ax)\,dx$$

let $ax = t \Rightarrow x = \frac{t}{a}$

$dx = \frac{dt}{a}$

If $x = -\infty \Rightarrow t = -\infty$

$x = \infty \Rightarrow t = \infty$

$$F[f(ax)] = \int_{-\infty}^{\infty} e^{i\alpha\left(\frac{t}{a}\right)} f(t)\,\frac{dt}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{i\alpha\left(\frac{t}{a}\right)} f(t)\,dt$$

$$F[f(ax)] = \frac{1}{a}F\left(\frac{\alpha}{a}\right)$$

Note:

1. $F_s[af(x)+bg(x)] = aF_s(\alpha) + bG_s(\alpha)$

$F_c[af(x)+bg(x)] = aF_c(\alpha) + bG_c(\alpha)$

2. $F_s[f(ax)] = \frac{1}{a}F_s\left(\frac{\alpha}{a}\right)$

$F_c[f(ax)] = \frac{1}{a}F_c\left(\frac{\alpha}{a}\right)$

3. Shifting property:-

If $F(\alpha)$ is the fourier transform of $f(x)$ then

$F[f(x-a)] = e^{i\alpha a}F(\alpha)$

Proof: By the definition $F(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$

$$F[f(x-a)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x-a)\,dx$$

$$= \int_{-\infty}^{\infty} e^{i\alpha(a+t)} f(t)\,dt$$

let $x-a=t$
$x = a+t$
$dx = dt$
if $x=-\infty \Rightarrow t=-\infty$
$x=\infty \Rightarrow t=\infty$

$$= \int_{-\infty}^{\infty} e^{i\alpha a} e^{i\alpha t} f(t)\,dt$$

$$= e^{i\alpha a} \int_{-\infty}^{\infty} e^{i\alpha t} f(t)\,dt \quad [\because \text{by definition}]$$

$$\therefore F[f(x-a)] = e^{i\alpha a} F(\alpha)$$

Modulation theorem :-

St: If $F(\alpha)$ is the fourier transform of $f(x)$ then $F[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a)+F(\alpha-a)]$

Proof: By the definition $F(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx \quad -①$

$$F[f(x)\cos ax] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \cos ax\,dx$$

$$= \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \left[\frac{e^{iax}+e^{-iax}}{2}\right] dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x) e^{i\alpha x} e^{iax}\,dx + \int_{-\infty}^{\infty} f(x) e^{i\alpha x} e^{-iax}\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x) e^{i(\alpha+a)x}\,dx + \int_{-\infty}^{\infty} f(x) e^{i(\alpha-a)x}\,dx\right]$$

$$F[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a)+F(\alpha-a)]$$

$$\text{lly } F[f(x)\sin ax] = \frac{1}{2i}[F(\alpha+a)-F(\alpha-a)]$$

Note :-

1. If $F_s(\alpha)$ be the fourier sine transform of $f(x)$ then

$$F_s[f(x)\cos ax] = \frac{1}{2}[F_s(\alpha+a)+F_s(\alpha-a)]$$

$$F_s[f(x)\sin ax] = \frac{1}{2}[F(\alpha-a)-F(\alpha+a)]$$

2. If $F_c(\alpha)$ be the fourier cosine transform of $f(x)$

then

$$F_c(f(x)\cos ax) = \frac{1}{2}[F_c(\alpha+a)+F_c(\alpha-a)]$$

$$F_c[f(x)\sin ax) = \frac{1}{2}[F_c(\alpha+a)-F_c(\alpha-a)]$$

1. Prove that $F[x^n \cdot f(x)] = (-i)^n \frac{d^n}{d\alpha^n}[F(\alpha)]$

By the definition of fourier transform

$$F(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx \quad -(1)$$

$$\frac{d}{d\alpha}[F(\alpha)] = \int_{-\infty}^{\infty} \frac{d}{d\alpha}(e^{i\alpha x}) f(x)\, dx$$

$$\int_{-\infty}^{\infty} (ix)\, e^{i\alpha x} f(x)\, dx \quad -(2)$$

$$\frac{d^2}{d\alpha^2}[F(\alpha)] = \frac{d}{d\alpha}\left[\frac{d}{d\alpha}[F(\alpha)]\right]$$

$$= \frac{d}{d\alpha}\left[\int_{-\infty}^{\infty} (ix)\, e^{i\alpha x} f(x)\, dx\right]$$

$$= \int_{-\infty}^{\infty} (ix) \frac{d}{d\alpha}(e^{i\alpha x}) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (ix)^2 e^{i\alpha x} f(x)\, dx$$

$$= (i)^2 \int_{-\infty}^{\infty} x^2 e^{i\alpha x} f(x)\, dx$$

$$\frac{d^2}{d\alpha^2}[F(\alpha)] = (i)^2 \int_{-\infty}^{\infty} e^{i\alpha x} [x^2 f(x)]\, dx$$

i.e., $\boxed{\frac{d^2}{d\alpha^2}[F(\alpha)] = (i)^2 F[x^2 f(x)]}$

lly $\frac{d^3}{d\alpha^3}[F(\alpha)] = (i)^3 F[x^3 f(x)]$

In general

$$\frac{d^n}{d\alpha^n}[F(\alpha)] = (i)^n F[x^n f(x)]$$

$$\therefore F[x^n f(x)] = (-i)^n \frac{d^n}{d\alpha^n}[F(\alpha)]$$

Scanned with CamScanner
Scanned with CamScanner

1. Find the fourier transform of $f(x) = x$ if $|x| \le a$
$= 0$ if $|x| > a$

Sol:- Given $f(x) = x$ if $|x| \le a$ i.e. $-a \le x \le a$ $\left[\begin{array}{c}|x|>a \\ x>a,\ x<-a\end{array}\right]$
$= 0$ if $|x| > a$ i.e. $-\infty < x < -a$ and $a < x < \infty$

The fourier transform of $f(x)$ is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{-a} f(x) e^{i\alpha x} dx + \int_{-a}^{a} f(x) e^{i\alpha x} dx + \int_{a}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x} dx + \int_{-a}^{a} x\, e^{i\alpha x} dx + \int_{a}^{\infty} 0 \cdot e^{i\alpha x} dx$$

$$= \int_{-a}^{a} x\, e^{i\alpha x}\, dx \qquad \left[\int uv = uv_1 - u' v_2 + \cdots\right]$$

$$= \left[x \frac{e^{i\alpha x}}{i\alpha} - (1)\frac{e^{i\alpha x}}{(i\alpha)^2}\right]_{-a}^{a} \qquad \left[\begin{array}{ll} u = x & v = e^{i\alpha x} \\ u' = 1, & v_1 = \frac{e^{i\alpha x}}{i\alpha},\ v_2 = \frac{e^{i\alpha x}}{(i\alpha)^2}\end{array}\right]$$

$$= \left[\left(\frac{a\, e^{i\alpha a}}{i\alpha} - \frac{e^{i\alpha a}}{i^2\alpha^2}\right) - \left(-a\frac{e^{-i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{i^2\alpha^2}\right)\right]$$

$$= \frac{a}{i\alpha} e^{i\alpha a} - \frac{e^{i\alpha a}}{(-1)\alpha^2} + \frac{a\, e^{-i\alpha a}}{i\alpha} + \frac{e^{-i\alpha a}}{(-1)\alpha^2}$$

$$= \frac{a}{i\alpha}\left[e^{i\alpha a} + e^{-i\alpha a}\right] + \frac{e^{i\alpha a}}{\alpha^2} - \frac{e^{-i\alpha a}}{\alpha^2}$$

$$= \frac{a}{i\alpha}(2\cos\alpha a) + \frac{1}{\alpha^2}\left[e^{i\alpha a} - e^{-i\alpha a}\right]$$

$$= \frac{2a\cos\alpha a}{i\alpha} + \frac{2i\sin\alpha a}{\alpha^2} \qquad \left[\begin{array}{l}\because e^{i\theta} + e^{-i\theta} = 2\cos\theta \\ e^{i\theta} - e^{-i\theta} = 2i\sin\theta\end{array}\right]$$

$$= \frac{2i\alpha a\cos\alpha a}{i^2\alpha^2} + \frac{2i\sin\alpha a}{\alpha^2}$$

$$= -\frac{2ai\alpha\cos\alpha a}{\alpha^2} + \frac{2i\sin\alpha a}{\alpha^2}$$

Hence $\boxed{F[f(x)] = \frac{2i}{\alpha^2}\left[\sin\alpha a - a\alpha\cos\alpha a\right]}$

Scanned with CamScanner
Scanned with CamScanner

2. Find the Fourier transform of $f(x) = \frac{1}{2a}$ for $|x| \le a$
$= 0$ for $|x| > a$, $a > 0$.

Sol:- The F.T. of $f(x)$ is

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x}\, dx + \int_{-a}^{a} \frac{1}{2a} e^{i\alpha x}\, dx + \int_{a}^{\infty} 0 \cdot e^{i\alpha x}\, dx$$

$$= \frac{1}{2a} \int_{-a}^{a} e^{i\alpha x}\, dx$$

$$= \frac{1}{2a} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^{a}$$

$$= \frac{1}{2a} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} \right] \qquad \left[\because e^{i\theta} - e^{-i\theta} = 2i \sin\theta \right]$$

$$= \frac{1}{\cancel{2}a} \left[\frac{\cancel{2i} \sin\alpha a}{\cancel{i}\alpha} \right]$$

$$\boxed{F[f(x)] = \frac{\sin\alpha a}{\alpha a}}$$

3. Find the Fourier transform of $f(x) = \cos x$ for $|x| \le a$
$= 0$ for $|x| > a$, $a > 0$.

Sol:- The F.T. of $f(x)$ is $F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$

ie. $$F[f(x)] = \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x}\, dx + \int_{-a}^{a} \cos x\, e^{i\alpha x}\, dx + \int_{a}^{\infty} 0 \cdot e^{i\alpha x}\, dx.$$

$$= \int_{-a}^{a} e^{i\alpha x} \cos x\, dx \qquad \left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b \sin bx) \right]$$

$$= \left[\frac{e^{i\alpha x}}{(i\alpha)^2 + 1^2} (i\alpha \cos x + \sin x) \right]_{-a}^{a}$$

$$= \frac{e^{i\alpha a}}{-\alpha^2+1} (i\alpha \cos a + \sin a) - \frac{e^{-i\alpha a}}{-\alpha^2+1} (i\alpha \cos a - \sin a) \Big]$$

$$= \frac{1}{1-\alpha^2} i\alpha \cos a \left[e^{i\alpha a} - e^{-i\alpha a} \right] + \frac{\sin a}{1-\alpha^2} \left[e^{i\alpha a} + e^{-i\alpha a} \right]$$

$$= \frac{1}{1-\alpha^2} i\alpha \cos a \,(2i \sin\alpha a) + \frac{\sin a}{1-\alpha^2} (2 \cos\alpha a) \qquad \left[\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2i \sin\theta \\ e^{i\theta} + e^{-i\theta} &= 2\cos\theta \end{aligned} \right]$$

$$= \frac{-2\alpha \cos a \sin\alpha a}{1-\alpha^2} + \frac{2 \sin a \cos\alpha a}{1-\alpha^2}$$

$$\therefore F[f(x)] = \frac{2}{1-\alpha^2} \left[\sin a \cos\alpha a - \alpha \cos a \sin\alpha a \right]$$

4. Find the Fourier transform of $f(x) = 1-x^2$, if $|x|<1$
$= 0$, if $|x|>1$

and use it to evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$.

Sol: Fourier transform of $f(x)$ is given by $\left[\because |x|<1, -1<x<1;\ |x|>1, x>1, x<-1\right]$

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{-1} 0\cdot e^{i\alpha x} dx + \int_{-1}^{1} (1-x^2) e^{i\alpha x} dx + \int_{1}^{\infty} (0)\, e^{i\alpha x} dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{i\alpha x}\, dx$$

$\left[\begin{array}{l} u = 1-x^2,\ v = e^{i\alpha x} \\ u' = -2x\ \ v_1 = \frac{e^{i\alpha x}}{i\alpha} \\ u'' = -2\ \ v_2 = \frac{e^{i\alpha x}}{(i\alpha)^2} \\ \int uv = uv_1 - u'v_2 + u''v_3 \end{array}\right]$

$$= \left[(1-x^2)\frac{e^{i\alpha x}}{i\alpha} - (-2x)\frac{e^{i\alpha x}}{(i\alpha)^2} + (-2)\frac{e^{i\alpha x}}{(i\alpha)^3}\right]_{-1}^{1}$$

$$= \left[\left(0\cdot\frac{e^{i\alpha}}{i\alpha} + 2(1)\frac{e^{i\alpha}}{(i\alpha)^2} - 2\frac{e^{i\alpha}}{(i\alpha)^3}\right) - \left(0\cdot\frac{e^{-i\alpha}}{i\alpha} + 2(-1)\frac{e^{-i\alpha}}{(i\alpha)^2} - 2\frac{e^{-i\alpha}}{(i\alpha)^3}\right)\right]$$

$$= \frac{2e^{i\alpha}}{i^2\alpha^2} - \frac{2e^{i\alpha}}{i^3\alpha^3} + \frac{2e^{-i\alpha}}{i^2\alpha^2} + \frac{2e^{-i\alpha}}{i^3\alpha^3}$$
$\left[\begin{array}{l} i^2 = -1 \\ i^3 = i^2 i = -i \end{array}\right]$

$$= -\frac{2e^{i\alpha}}{\alpha^2} - \frac{2e^{i\alpha}}{(-i)\alpha^3} + \frac{2e^{-i\alpha}}{(-1)\alpha^2} + \frac{2e^{-i\alpha}}{(-i)\alpha^3}$$

$$= -\frac{2}{\alpha^2}\left[e^{i\alpha} + e^{-i\alpha}\right] + \frac{2}{i\alpha^3}\left[e^{i\alpha} - e^{-i\alpha}\right]$$

$$= -\frac{2}{\alpha^2}(2\cos\alpha) + \frac{2}{\cancel{i}\alpha^3}(2\cancel{i}\sin\alpha)$$
$\left[\begin{array}{l} e^{i\theta}+e^{-i\theta} = 2\cos\theta \\ e^{i\theta}-e^{-i\theta} = 2i\sin\theta \end{array}\right]$

$$= \frac{-4}{\alpha^2}\cos\alpha + \frac{4\sin\alpha}{\alpha^3}$$

$$F[f(x)] = -\frac{4}{\alpha^3}\left[\alpha\cos\alpha - \sin\alpha\right] = F(\alpha) \quad \text{--- (1)}$$

By Inversion formula for Fourier transform, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha$

i.e. $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{\alpha^3}\left[\alpha\cos\alpha - \sin\alpha\right](\cos\alpha x - i\sin\alpha x)\, d\alpha$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\alpha\cos\alpha - \sin\alpha)\cos\alpha x}{\alpha^3}\, d\alpha + \frac{2i}{\pi}\int_{-\infty}^{\infty}(\alpha\cos\alpha - \sin\alpha)\sin\alpha x\, d\alpha$$

$$= \frac{-4}{\pi}\int_0^\infty \frac{(\alpha\cos\alpha - \sin\alpha)\cos\alpha x}{\alpha^3}\, d\alpha$$

[$\because$ the integrand in the 1st integral is even and that in the second integral is odd]

$$\Rightarrow -\frac{4}{\pi}\int_0^\infty \left(\frac{\alpha\cos\alpha-\sin\alpha}{\alpha^3}\right)\cos\alpha x\, d\alpha = f(x)$$

$$\Rightarrow \int_0^\infty \left(\frac{\alpha\cos\alpha-\sin\alpha}{\alpha^3}\right)\cos\alpha x\, d\alpha = -\frac{\pi}{4}f(x) = -\frac{\pi}{4}(1-x^2), \text{ if } |x|<1$$

$$= 0, \quad \text{if } |x|>1$$

Putting $x=\frac{1}{2}$, we have $\int_0^\infty \frac{(\alpha\cos\alpha-\sin\alpha)}{\alpha^3}\cos\left(\frac{\alpha}{2}\right)d\alpha = -\frac{\pi}{4}\left(1-\frac{1}{4}\right)$

$$= \frac{-3\pi}{16}$$

Hence, $\boxed{\int_0^\infty \frac{(x\cos x-\sin x)}{x^3}\cos\left(\frac{x}{2}\right)dx = -\frac{3\pi}{16}.}$ $\left[\int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$

Note:- If the function $f(x)$ is equal to $F(\alpha)$ then $f(x)$ is called Self-reciprocal.

5. Show that the Fourier transform of $e^{-x^2/2}$ is ~~self reciprocal~~ self reciprocal

Let $f(x) = e^{-x^2/2}$ then

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$\left[f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt\right]e^{-i\alpha x}d\alpha \right.$$

$$\left. = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt\right]e^{-i\alpha x}d\alpha \right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[x^2-2i\alpha x+(i\alpha)^2-(i\alpha)^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-i\alpha)^2-i^2\alpha^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-i\alpha)^2+\alpha^2]}\, dx$$

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx\right.$$

$$= \cancel{2}\int_0^\infty e^{-t}\frac{1}{\cancel{2}}t^{-\frac{1}{2}}dt \quad x^2=t$$

$$x=\sqrt{t}$$

$$= \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt \quad dx = \frac{1}{2}t^{-\frac{1}{2}}dt$$

$$= \Gamma\left(\tfrac{1}{2}\right)$$

$$\left. = \sqrt{\pi} \right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x-i\alpha}{\sqrt{2}}\right)^2} e^{-\alpha^2/2}\, dx \quad \text{put } \frac{x-i\alpha}{\sqrt{2}} = t$$

$$\Rightarrow dx = \sqrt{2}\,dt$$

As $x\to-\infty$, $t\to-\infty$

as $x\to\infty$, $t\to\infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\, e^{-\alpha^2/2}\, \sqrt{2}\, dt$$

$$= \frac{e^{-\alpha^2/2}\sqrt{2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}\right]$$

$$F[f(x)] = \frac{\cancel{\sqrt{2}}\, e^{-\alpha^2/2}}{\cancel{\sqrt{2\pi}}}\cancel{\sqrt{\pi}} = e^{-\alpha^2/2} \text{ (or) } e^{-x^2/2}$$

$\therefore$ $f(x)$ is self-reciprocal.

6. Show that the Fourier transform of $f(x) = a-|x|$, for $|x|<a$
$= 0$, for $|x|>a>0$

is $\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\alpha}{\alpha^2}\right]$. Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Sol:- The Fourier transform of $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\cdot e^{i\alpha x}\,dx + \int_{-a}^{a}(a-|x|)e^{i\alpha x}\,dx + \int_{a}^{\infty} 0\cdot e^{i\alpha x}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} a\, e^{i\alpha x}\,dx - \int_{-a}^{a}|x|\, e^{i\alpha x}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\left[\frac{e^{i\alpha x}}{i\alpha}\right]_{-a}^{a} - \left[\int_{-a}^{0} -x\, e^{i\alpha x}\,dx + \int_{0}^{a} x\, e^{i\alpha x}\,dx\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{i\alpha}\left(e^{i\alpha a} - e^{-i\alpha a}\right) + \left[x\frac{e^{i\alpha x}}{i\alpha} - (1)\frac{e^{i\alpha x}}{(i\alpha)^2}\right]_{-a}^{0} - \left[x\frac{e^{i\alpha x}}{i\alpha} - \frac{e^{i\alpha x}}{(i\alpha)^2}\right]_{0}^{a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{i\alpha}e^{i\alpha a} - \frac{a}{i\alpha}e^{-i\alpha a} + \left[\left(0 - \frac{e^{0}}{-\alpha^2}\right) - \left(-a\frac{e^{-i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{-\alpha^2}\right)\right]\right.$$
$$\left. - \left[\left(\frac{a\,e^{i\alpha a}}{i\alpha} - \frac{e^{i\alpha a}}{-\alpha^2}\right) - \left(0 - \frac{e^{0}}{-\alpha^2}\right)\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{\frac{a}{i\alpha}e^{i\alpha a}} - \cancel{\frac{a}{i\alpha}e^{-i\alpha a}} + \frac{1}{\alpha^2} + \cancel{\frac{a\,e^{-i\alpha a}}{i\alpha}} + \frac{e^{-i\alpha a}}{-\alpha^2} - \cancel{\frac{a\,e^{i\alpha a}}{i\alpha}} + \frac{e^{i\alpha a}}{-\alpha^2} + \frac{1}{\alpha^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\alpha^2} - \frac{1}{\alpha^2}\left(e^{i\alpha a} + e^{-i\alpha a}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\alpha^2} - \frac{1}{\alpha^2}\, 2\cos\alpha a\right] \qquad \left[\because e^{i\theta} + e^{-i\theta} = 2\cos\theta\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1-\cos a\alpha}{\alpha^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\alpha}{\alpha^2}\right]$$

$$\boxed{\therefore\ F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\alpha}{\alpha^2}\right]}$$

Deduction: The inverse Fourier transform of $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos a\alpha}{\alpha^2}\right] (\cos\alpha x - i\sin\alpha x)\, d\alpha.$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \left(\frac{1-\cos a\alpha}{\alpha^2}\right)\cos\alpha x\, d\alpha - i\int_{-\infty}^{\infty}\left(\frac{1-\cos a\alpha}{\alpha^2}\right)\sin\alpha x\, d\alpha\right]$$

$$= \frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos a\alpha}{\alpha^2}\right)\cos\alpha x\, d\alpha \qquad \left[\begin{array}{ll} \text{1st integral Integrand} & = \text{even} \\ \text{2nd} \quad '' \quad '' & = \text{odd}\end{array}\right]$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos a\alpha}{\alpha^2}\right)\cos\alpha x\, d\alpha = f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a. \end{cases}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{1-\cos a\alpha}{\alpha^2}\right)\cos\alpha x\, d\alpha = \begin{cases} \frac{\pi}{2}(a-|x|), & |x|<a \\ 0, & |x|>a. \end{cases}$$

put $a=2,\ x=0$

$$\therefore \int_0^{\infty}\frac{(1-\cos 2\alpha)}{\alpha^2}\cos 0\, d\alpha = \frac{\pi}{2}[2-0] = \frac{2\pi}{2}$$

$$\Rightarrow \int_0^{\infty}\frac{2\sin^2\alpha}{\alpha^2}(1)\, d\alpha = \frac{\cancel{2}\pi}{\cancel{2}}.$$

$$\Rightarrow \int_0^{\infty}\frac{\sin^2\alpha}{\alpha^2}\, d\alpha = \frac{\pi}{2}$$

$$\Rightarrow \boxed{\int_0^{\infty}\left(\frac{\sin\alpha}{\alpha}\right)^2 d\alpha = \frac{\pi}{2}.}$$

(or) $$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

H.W 1) Find the Fourier transform of $f(x)$ defined by $f(x) = 1,\ |x|<a$; $= 0,\ |x|>a$

and hence evaluate i) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin a\alpha\cos\alpha x}{\alpha}\, d\alpha$ ii) $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

7. Find the Fourier transform of $e^{-a|x|}$ $(a>0)$ and hence show that

i) $\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp = \frac{\pi}{2a} e^{-a|x|}$ ii) $F\left[x\, e^{-a|x|}\right] = \frac{i4ap}{(a^2+p^2)^2}$.

Sol:- The fourier transform of $f(x)$ is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx \qquad \left[\because \alpha = p\right]$$

Let $f(x) = e^{-a|x|}$ then

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a|x|}\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a|x|}(\cos px + i\sin px)\,dx \qquad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= \int_{-\infty}^{\infty} e^{-a|x|}\cos px\,dx + i\int_{-\infty}^{\infty} e^{-a|x|}\sin px\,dx$$

$$= 2\int_0^\infty e^{-a|x|}\cos px\,dx + i(0) \qquad \left[\because \text{Integrand in } 1^{st} \text{ integral is even and odd in } 2^{nd} \text{ integral}\right]$$

$$= 2\int_0^\infty e^{-ax}\cos px\,dx \qquad \left[\because |x| = x \text{ in } (0,\infty)\right]$$

$$= 2\frac{a}{a^2+p^2} = \frac{2a}{a^2+p^2} = F(p) \qquad \left[\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\right] \quad \text{—(1)}$$

Deduction: i) By inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp \qquad \left[\because \alpha = p\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+p^2}\, e^{-ipx}\, dp \qquad \left[\because (1)\right]$$

$$= \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}a}{a^2+p^2}(\cos px - i\sin px)\,dp \qquad \left[e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos px}{a^2+p^2}\,dp - i\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\sin px}{a^2+p^2}\,dp$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp - i(0) \qquad \left[\because \text{Integrand in the } 2^{nd} \text{ integral is odd}\right]$$

$$\Rightarrow f(x) = \frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp \qquad \left[\begin{array}{l}\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \\ \text{when } f(x) \text{ is even} \\ \& \int_{-a}^{a} f(x)\,dx = 0, \text{ when } f(x) \text{ is odd}\end{array}\right]$$

$$\Rightarrow \frac{\pi}{2a} f(x) = \int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$$

$$\boxed{\therefore \int_0^\infty \frac{\cos px}{a^2+p^2}\,dp = \frac{\pi}{2a} e^{-a|x|}}$$

ii) w.r.t. $F[x^n f(x)] = (-i)^n \frac{d^n}{dp^n}[F(p)]$ $[\because \alpha = p]$

For $n=1$, $F[x f(x)] = -i \frac{d}{dp}[F(p)]$

$$\therefore F[x e^{-a|x|}] = -i \frac{d}{dp}\left[\frac{2a}{a^2+p^2}\right] \quad [\because ①]$$

$$= -i(2a)\frac{d}{dp}\left[\frac{1}{a^2+p^2}\right]$$

$$= -i(2a)\left[-\frac{1}{(a^2+p^2)^2}(0+2p)\right]$$

$$= \frac{i4ap}{(a^2+p^2)^2}$$

8. Find the fourier transform of e^{-x^2}. Hence find the Fourier transform of i) e^{-ax^2}, $(a>0)$ ii) $e^{-x^2/2}$ iii) $e^{-4(x-3)^2}$ iv) $e^{-x^2}\cos 2x$.

Sol:- The Fourier transform of $f(x) = e^{-x^2}$ is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2+i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[x^2-i\alpha x]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[x^2-2x\left(\frac{i\alpha}{2}\right)+\left(\frac{i\alpha}{2}\right)^2-\left(\frac{i\alpha}{2}\right)^2]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[(x-i\alpha/2)^2]+\left(\frac{i\alpha}{2}\right)^2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-[x-\frac{i\alpha}{2}]^2} e^{\frac{i^2\alpha^2}{4}}\, dx \quad (i^2=-1)$$

put $x - \frac{i\alpha}{2} = t$

$\Rightarrow dx = dt$

As $x \to -\infty$, $t \to -\infty$

As $x \to \infty$, $t \to \infty$

$$= \int_{-\infty}^{\infty} e^{-t^2} e^{-\frac{\alpha^2}{4}}\, dt$$

$$= e^{-\alpha^2/4}\int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= 2 e^{-\alpha^2/4}\int_0^{\infty} e^{-t^2}\, dt$$

$$F[f(x)] = 2 e^{-\alpha^2/4}\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\, e^{-\frac{\alpha^2}{4}}$$

$$\therefore F[f(x)] = \sqrt{\pi}\, e^{-\alpha^2/4} = F(\alpha) \quad — ①$$

$\Big[\because \int_0^{\infty} e^{-t^2}\, dt$, $t^2 = z$, $t = \sqrt{z}$, $dt = \frac{1}{2} z^{-\frac{1}{2}}\, dz$

$= \int_0^{\infty} e^{-z} \frac{1}{z^{\frac{1}{2}}} \frac{dz}{2}$

$= \frac{1}{2}\int_0^{\infty} e^{-z} z^{\frac{1}{2}-1}\, dz$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \Big]$

8. i) $e^{-ax^2} = e^{-(\sqrt{a}x)^2} = f(\sqrt{a}x)$

By change of scale property, we have

$$F[f(ax)] = \frac{1}{a} F\left[\frac{\alpha}{a}\right]$$

$$\Rightarrow F\left[e^{-ax^2}\right] = F\left[e^{-(\sqrt{a}x)^2}\right]$$

$$= \frac{1}{\sqrt{a}} \sqrt{\pi}\, e^{-\frac{1}{4}\left(\frac{\alpha}{\sqrt{a}}\right)^2} = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{\alpha^2}{4a}}$$

ii) Put $a = \frac{1}{2}$ in (i), we have

$$F\left[e^{-x^2/2}\right] = \sqrt{\frac{\pi}{\frac{1}{2}}}\, e^{-\frac{\alpha^2}{4(\frac{1}{2})}} = \sqrt{2\pi}\, e^{-\alpha^2/2}.$$

iii) $e^{-4x^2} = e^{-(2x)^2} = f(2x)$

By change of scale property, we have

$$F[f(2x)] = \frac{1}{2} F\left[\frac{\alpha}{2}\right] \qquad \left[\because F(\alpha) = \sqrt{\pi}\, e^{-\alpha^2/4}\right]$$

$$= \frac{1}{2} \sqrt{\pi}\, e^{-\frac{1}{4}\left[\frac{\alpha}{2}\right]^2}$$

$$= \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{16}} \quad \text{— I}$$

By shifting property $F[f(x-a)] = e^{i\alpha a} F(\alpha)$

$$\therefore F\left[e^{-4(x-3)^2}\right] = e^{i\alpha 3} \frac{\sqrt{\pi}}{2} e^{-\alpha^2/16} \qquad \left[\text{Here } F(\alpha) \text{ is the F.T. of } f(2x)\right]$$

$$= \frac{\sqrt{\pi}}{2} e^{3i\alpha - \frac{\alpha^2}{16}}.$$

iv) By Modulation theorem

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(\alpha+a) + F(\alpha-a)\right]$$

$$F\left[e^{-x^2}\cos 2x\right] = \frac{1}{2}\left[\sqrt{\pi}\, e^{-\frac{1}{4}(\alpha+2)^2} + \sqrt{\pi}\, e^{-\frac{1}{4}(\alpha-2)^2}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-\frac{1}{4}(\alpha+2)^2} + e^{-\frac{1}{4}(\alpha-2)^2}\right]$$

__H.W.__ 1) Find the fourier transform of $f(x) = a^2 - x^2, \ |x| < 1$

$= 0, \quad |x| > 1$

Deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. [Hint for deduction taking $a = 1$ & $x = 0$]

14/7/14

Q. Find the fourier sine and cosine transform $2e^{-5x}+5e^{-2x}$.

Sol:- let $f(x) = 2e^{-5x}+5e^{-2x}$.

The fourier sine transform of $f(x)$ is given by

$$F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x\, dx$$

$$= \int_0^\infty \left(2e^{-5x}+5e^{-2x}\right)\sin\alpha x\, dx$$

$$= 2\int_0^\infty e^{-5x}\sin\alpha x\, dx + 5\int_0^\infty e^{-2x}\sin\alpha x\, dx$$

$$= 2\left[\frac{e^{-5x}}{\alpha^2+25}(-5\sin\alpha x - \alpha\cos\alpha x)\right]_0^\infty + 5\left[\frac{e^{-2x}}{\alpha^2+4}(-2\sin\alpha x - \alpha\cos\alpha x)\right]_0^\infty$$

$$= \frac{2}{\alpha^2+25}\left[0 - e^{-0}(-5(0)-\alpha(1))\right] + \frac{5}{\alpha^2+4}\left[0 - e^{-0}(-2(0)-\alpha(1))\right]$$

$$= \frac{2\alpha}{\alpha^2+25} + \frac{5\alpha}{\alpha^2+4} \quad \left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

To find fourier cosine transform.

the fourier cosine transform of $f(x)$ is

$$F_c(\alpha) = \int_0^\infty f(x)\cos\alpha x \, dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\cos\alpha x \, dx$$

$$= 2\int_0^\infty e^{-5x}\cos\alpha x \, dx + 5\int_0^\infty e^{-2x}\cos\alpha x \, dx$$

$$= 2\left[\frac{e^{-5x}}{\alpha^2+5^2}(-5\cos\alpha x + \alpha\sin\alpha x)\right]_0^\infty + 5\left[\frac{e^{-2x}}{\alpha^2+4}(-2\cos\alpha x + \alpha\sin\alpha x)\right]_0^\infty$$

$$= 2\left[0 - \frac{1}{\alpha^2+25}(-5(1)+0)\right] + \frac{5}{\alpha^2+4}\left(0 - 1(-2(1)+0)\right)$$

$$= \frac{10}{\alpha^2+25} + \frac{10}{\alpha^2+4} \quad \left[\because \int e^{ax}\cos bx \, dx = \frac{(a\cos bx + b\sin bx)e^{ax}}{a^2+b^2}\right]$$

Q.10. Find the fourier sine transform $e^{-|x|}$ and hence ST $\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}, \; m>0$

Sol:- let $f(x) = e^{-|x|}$

fourier sine transform of $f(x)$ is

$$F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x \, dx$$

$$= \int_0^\infty e^{-x}\sin\alpha x \, dx$$

$$= \frac{e^{-x}}{1^2+\alpha^2}\left[-\sin\alpha x - \alpha\cos\alpha x\right]_0^\infty$$

$$= \frac{1}{\alpha^2+1}\left[0 - e^{-0}(-\alpha(1))\right]$$

$$F_s(\alpha) = \frac{\alpha}{\alpha^2+1}$$

F.S. Deduction:

By the inverse fourier sine transform of $F_s(\alpha)$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\alpha)\sin\alpha x \, d\alpha$$

$$\therefore e^{-|x|} = \frac{2}{\pi}\int_0^{\infty} \frac{\alpha}{\alpha^2+1}\sin\alpha x \, d\alpha$$

changing x to m

$$e^{-|m|} = \frac{2}{\pi}\int_0^{\infty} \frac{\alpha}{\alpha^2+1}\sin m\alpha \, d\alpha$$

$$\int_0^{\infty} \frac{\alpha}{\alpha^2+1}\sin m\alpha \, d\alpha = \frac{\pi}{2}e^{-|m|}$$

$$= \frac{\pi}{2}e^{-m}, \quad m>0$$

(or) $$\int_0^{\infty} \frac{x}{x^2+1}\sin mx \, dx = \frac{\pi}{2}e^{-m}$$

11. Find the fourier cosine transform of $f(x) = x$ for $0<x<1$

$= 2-x$ for $1<x<2$

$= 0$ for $x>2$

Sol:- The fourier cosine transform of $f(x)$ is given by

$$F_c(\alpha) = \int_0^{\infty} f(x)\cos\alpha x \, dx$$

$$= \int_0^1 x\cos\alpha x\,dx + \int_1^2 (2-x)\cos\alpha x\,dx + \int_2^{\infty} 0\,dx$$

$$= \left[x\cdot\frac{\sin\alpha x}{\alpha} + \frac{\cos\alpha x}{\alpha^2}\right]_0^1 + \left[(2-x)\frac{\sin\alpha x}{\alpha} + (-1)\frac{\cos\alpha x}{\alpha^2}\right]_1^2$$

$$= \left[\frac{1}{\alpha}\sin\alpha + \frac{\cos\alpha}{\alpha^2} - \left(0+\frac{1}{\alpha^2}\right)\right] + \left[0 - \frac{1}{\alpha^2}\cos 2\alpha - \left(\frac{\sin\alpha}{\alpha} - \frac{\cos\alpha}{\alpha^2}\right)\right]$$

$$\therefore \cancel{\frac{\sin\alpha}{\alpha}} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2} - \frac{\cos 2\alpha}{\alpha^2} - \cancel{\frac{\sin\alpha}{\alpha}} + \frac{\cos\alpha}{\alpha^2}$$

$$\therefore \frac{2\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2} - \frac{\cos 2\alpha}{\alpha^2} = \frac{1}{\alpha^2}\left[2\cos\alpha - \cos 2\alpha - 1\right]$$

$$= \frac{1}{\alpha^2}\left[2\cos\alpha - (2\cos^2\alpha - 1) - 1\right]$$

$$= \frac{2\cos\alpha - 2\cos^2\alpha}{\alpha^2} = \frac{2\cos\alpha(1-\cos\alpha)}{\alpha^2}$$

42. Find the fourier sine transform of $\frac{e^{-ax}}{x}$

Sol:- let $f(x) = \frac{e^{-ax}}{x}$

$$F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x\, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x}\sin\alpha x\, dx \quad \text{—} ①$$

diff ① w.r.t α

$$\frac{d}{d\alpha}[F_s(\alpha)] = \int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cos\alpha x \cdot \cancel{x}\, dx$$

$$= \int_0^\infty e^{-ax}\cos\alpha x\, dx$$

$$\frac{d}{d\alpha}[F_s(\alpha)] = \frac{a}{a^2+\alpha^2} \qquad \left[\because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}\right]$$

Integrating on b.side

$$F_s(\alpha) = \int \frac{a}{a^2+\alpha^2}\, d\alpha$$

$$= \cancel{a} \cdot \alpha \cdot \frac{1}{\cancel{\alpha}}\tan^{-1}\left(\frac{\alpha}{a}\right) + c$$

$$F_s(\alpha) = \tan^{-1}\left(\frac{\alpha}{a}\right) + c$$

If $\alpha = 0$ the $F_s(\alpha) = 0$

$$0 = \tan^{-1}(0) + c$$

$$0 = 0 + c \Rightarrow c = 0$$

$$\boxed{\therefore F_s(\alpha) = \tan^{-1}\left(\frac{\alpha}{a}\right)}$$

13/7/19

43. Find the fourier sine and cosine transform of x^{n-1}, $n > 0$

Sol:- let $f(x) = x^{n-1}$

then $F_s[f(x)] = x^{n-1}$

then $F_s[f(x)] = \int_0^\infty f(x)\sin\alpha x\, dx$

Scanned with CamScanner

Scanned with CamScanner

Scanned with CamScanner

$$F_s[x^{n-1}] = \int_0^\infty x^{n-1} \sin\alpha x \, dx \quad -①$$

$$F_c[xf(x)] = \int_0^\infty f(x) \cos\alpha x$$

$$F_c[x^{n-1}] = \int_0^\infty x^{n-1} \cos\alpha x \, dx \quad -②$$

② + i ①

we get $$F_c(x^{n-1}) + iF_s(x^{n-1}) = \int_0^\infty x^{n-1} \cos\alpha x \, dx + i\int_0^\infty x^{n-1} \sin\alpha x \, dx$$

$$= \int_0^\infty x^{n-1} [\cos\alpha x + i\sin\alpha x] \, dx$$

$$= \int_0^\infty x^{n-1} e^{i\alpha x} \, dx$$

let $i\alpha x = -t$

$x = \frac{-t}{i\alpha}$

$x = \left(\frac{i}{\alpha}\right) t$

$dx = \frac{i}{\alpha} dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$= \int_0^\infty e^{i\alpha x} x^{n-1} \, dx$$

$$= \int_0^\infty e^{-t} \left(\frac{i}{\alpha} t\right)^{n-1} \left(\frac{i}{\alpha}\right) dt$$

$$= \left(\frac{i}{\alpha}\right)^n \int_0^\infty e^{-t} t^{n-1} \, dt$$

$$= \left(\frac{i}{\alpha}\right)^n \Gamma(n)$$

$$= \frac{1}{\alpha^n} \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^n \Gamma(n)$$

By Demoiver's theorem
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$$F_c(x^{n-1}) + iF_s(x^{n-1}) = \frac{\Gamma(n)}{\alpha^n}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]$$

comparing the real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{\alpha^n} \cos\frac{n\pi}{2}$$

$$F_s(x^{n-1}) = \frac{\Gamma(n)}{\alpha^n} \sin\frac{n\pi}{2}.$$

Q14. Find the fourier cosine transform of e^{-x^2} then

Sol:- let $f(x) = e^{-x^2}$

$$F_c[f(x)] = \int_0^\infty f(x) \cos\alpha x \, dx$$

$$= \int_0^\infty e^{-x^2} \cos\alpha x \, dx$$

Scanned with CamScanner
Scanned with CamScanner

let $f(x) = e^{-x^2}$

$$F_c[f(x)] = \int_0^\infty f(x) \cos \alpha x \, dx$$

$$= \int_0^\infty e^{-x^2} \cos \alpha x \, dx = I \quad \text{— (1) say}$$

Diff w.r.t 'α' we get

$$\frac{dI}{d\alpha} = \int_0^\infty e^{-x^2} (-x \sin \alpha x) \, dx$$

$$= \frac{1}{2} \int_0^\infty (-2x) e^{-x^2} \sin \alpha x \, dx$$

$$= \frac{1}{2} \int_0^\infty d(e^{-x^2}) \sin \alpha x \, dx$$

Scanned with CamScanner
Scanned with CamScanner

$$= \frac{1}{2}\left[\left(\sin\alpha x\, e^{-x^2}\right)_0^\infty - \int_0^\infty \alpha\cos\alpha x\, e^{-x^2}\,dx\right]$$

$$= \frac{1}{2}(0-0) - \frac{\alpha}{2}\underbrace{\int_0^\infty e^{-x^2}\cos\alpha x\,dx}_{I} \quad \text{- from ①}$$

$$\frac{dI}{d\alpha} = -\frac{\alpha}{2} I$$

$$\frac{dI}{I} = -\frac{\alpha}{2}\,d\alpha$$

$$\log I = -\frac{\alpha^2}{4} + \log c$$

$$\log\frac{I}{c} = \frac{-\alpha^2}{4}$$

$$\frac{I}{c} = e^{-\alpha^2/4}$$

$$I = c e^{-\alpha^2/4} \quad - ②$$

If $\alpha = 0$ then $I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

$$\frac{\sqrt{\pi}}{2} = c \cdot e^{-0} \Rightarrow c = \frac{\sqrt{\pi}}{2}.$$

Sub $c = \frac{\sqrt{\pi}}{2}$ in ②

$$I = F_c\{f(x)\} = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}.$$

#15. Find the fourier ~~cosfor~~ cosine transform of

$$f(x) = \frac{1}{1+x^2}.$$

Sol: let $f(x) = \frac{1}{1+x^2}$

$$F_c\left[f(x)\right] = \int_0^\infty f(x)\cos\alpha x\,dx$$

$$= \int_0^\infty \frac{1}{1+x^2}\cos\alpha x\,dx = I \text{ (say)} \quad - ①$$

Diff ① w.r.t α

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{-x\sin\alpha x}{1+x^2}\,dx$$

$$= -\int_0^\infty \frac{x^2\sin\alpha x}{x(1+x^2)}\,dx$$

$$= -\int_0^\infty \frac{(1+x^2-1)\sin\alpha x}{x(1+x^2)}\,dx$$

$$= -\int_0^\infty \frac{\sin\alpha x}{x}\,dx + \int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx$$

Scanned with CamScanner

Scanned with CamScanner

$$= \frac{\pi}{2} + \int_0^\infty \frac{\sin \alpha x}{x(1+x^2)}\, dx \quad \text{—} (2)$$

Diff (2) w.r.t 'α' then

$$\frac{d^2 I}{d\alpha^2} = 0 + \int_0^\infty \frac{x \cos \alpha x}{x(1+x^2)}\, dx$$

$$\frac{d^2 I}{d\alpha^2} = I$$

$$D^2 I - I = 0$$

$$(D^2 - 1) I = 0 \quad \text{—} (3)$$

The General Solution of (3) is

$$I = c_1 e^{-\alpha} + c_2 e^{\alpha} \quad \text{—} (4)$$

If $\alpha = 0$ then $I = \int_0^\infty \frac{1}{1+x^2}\, dx$

$$= (\tan^{-1} x)_0^\infty = \frac{\pi}{2}.$$

i.e $\frac{\pi}{2} = c_1 + c_2 \quad \text{—} (5)$

Diff (4) w.r.t α we get

$$\frac{dI}{d\alpha} = -c_1 e^{-\alpha} + c_2 e^{\alpha}$$

If $\alpha = 0$ then $\frac{dI}{d\alpha} = 0$

$$0 = -c_1 + c_2 \quad \text{—} (6)$$

Solving (5) & (6)

$$2c_2 = \frac{\pi}{2} \Rightarrow c_2 = \frac{\pi}{4}$$

$$\Rightarrow c_1 = \frac{\pi}{4}$$

$$I = \frac{\pi}{4} e^{-\alpha} + \frac{\pi}{4} e^{\alpha}$$

$$= \frac{\pi}{4}\left(e^{\alpha} + e^{-\alpha}\right) = \frac{\pi}{2} \cosh \alpha$$

16/7/14

Finite fourier sine transforms

The finite let $f(x)$ be a function defined in $0<x<l$ then the finite fourier sine transform of $f(x)$ is defined by

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

and the function

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l}$$

is called the inverse finite fourier sine transform of $F_s(n)$

where 'n' is an integer.

Finite fourier cosine transform :-

let $f(x)$ be a function defined in $0<x<l$, then the fourier cosine transform of $f(x)$ is defined by

$$F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

and the function $f(x)$ is given by

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l}$$

is called inverse finite fourier cosine transform of $F_c(n)$

1. Find the finite fourier sine and cosine transform $f(x) = x$ where $0<x<4$

Sol:- Given $f(x) = x$

To find finite fourier sine transform

The finite fourier sine transform of $f(x)$ in $(0,l)$ is given by

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \quad [\because l = 4]$$

$$= \int_0^4 x \sin \frac{n\pi x}{4}\, dx$$

$$\left[\begin{array}{l} \int uv \\ = uv_1 - u'v_2 + \text{—} \\ u = x,\ v = \sin\frac{n\pi x}{4} \end{array}\right.$$

$$= \left(x \cdot \frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}} + 1 \frac{\sin\frac{n\pi x}{4}}{\left(\frac{n\pi}{4}\right)^2} \right)_0^4$$

$$= \left[-4 \cdot \frac{4}{n\pi} \cos n\pi - 0 + \left(\frac{4}{n\pi}\right)^2 \sin n\pi - 0 \right]$$

$$= -\frac{16}{n\pi}(-1)^n$$

$$\boxed{F_s(n) = \frac{16}{n\pi}(-1)^{n+1}}$$

To find finite fourier cosine transform:

The finite cosine transform of $f(x)$ in $(0,l)$ is given by $F_c(n) = \int_0^l f(x) \cos\frac{n\pi x}{l} dx$

$$= \int_0^4 x \cos\frac{n\pi x}{4} dx \qquad \left[\begin{array}{l} \int uv = uv_1 - u'v_2 + \cdots \\ u = x,\ v = \cos\frac{n\pi}{4}x \end{array} \right.$$

$$= \left| x \cdot \frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}} + (1)\frac{\cos\frac{n\pi x}{4}}{\left(\frac{n\pi}{4}\right)^2} \right|_0^4$$

$$= \left[4 \cdot \frac{4}{n\pi} \sin n\pi - 0\right] + \left(\frac{4}{n\pi}\right)^2 \left[\cos n\pi - \cos 0\right]$$

$$= 0 + \left(\frac{4}{n\pi}\right)^2 \left[(-1)^n - 1\right]$$

$$F_c(n) = \frac{16}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad n > 0$$

If $n = 0$ then $F_c(0) = \int_0^4 x\, dx$

$$= \left(\frac{x^2}{2}\right)_0^4 = 8$$

$$\therefore F_c(n) = \frac{16}{n^2\pi^2}\left[(-1)^n - 1\right] \text{ if } n > 0 \quad n \neq 0$$

$$= 8 \text{ if } n = 0$$

2. Find the finite fourier cosine transform of $\left(1 - \frac{x}{\pi}\right)^2$ in $(0,\pi)$.

Sol: Given $f(x) = \left(1 - \frac{x}{\pi}\right)^2$

The finite fourier cosine transform $f(x)$ in $(0,\pi)$ is given by $F_c(n) = \int_0^\pi f(x) \cos nx\, dx$

$$= \int_0^\pi \left(1 - \frac{x}{\pi}\right)^2 \cos nx\, dx$$

$$= \left[\left(1-\frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - 2\left(1-\frac{x}{\pi}\right)\left(\frac{-1}{\pi}\right) \frac{-\cos nx}{n^2} + 2\left(\frac{-1}{\pi}\right)\left(\frac{-1}{\pi}\right) \frac{-\sin nx}{n^3} \right]_0^\pi$$

$$= \left[\left(1-\frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - \frac{2}{n^2\pi}\left(1-\frac{x}{\pi}\right)\cos nx - \frac{2}{n^3\pi^2} \sin nx \right]_0^\pi$$

$$= \left[0 - 0 - \frac{2}{n^3\pi^2} \sin n\pi - \left[\frac{\sin 0}{n} - \frac{2}{n^2\pi} \cos 0 - \frac{2}{n^3\pi^2} \sin 0 \right] \right]$$

$F_c(n) = \frac{2}{n^2\pi}$, if $n > 0$

If $n = 0$ $\quad F_c(0) = \int_0^\pi f(x) \cos dx$

$$= \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 dx = -\pi \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 \left(-\frac{1}{\pi}\right) dx$$

$$= -\pi \left| \frac{\left(1-\frac{x}{\pi}\right)^3}{3} \right|_0^\pi \quad \left[\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= -\frac{\pi}{3} \cdot (0-1) = \frac{\pi}{3}$$

$F_c(n) = \frac{2}{n^2\pi}$ if $n>0$

$= \frac{\pi}{3}$ if $n=0$

3\. Show that finite fourier sine transform of $\frac{x}{\pi}$ is $\frac{(-1)^{n+1}}{n}$ in the interval $(0,\pi)$

Sol: Given $f(x) = \frac{x}{\pi}$.

The finite fourier sine transform of $f(x)$ in $(0,\pi)$ is

$$F_s(n) = \int_0^\pi f(x) \sin nx\, dx$$

$$= \int_0^\pi \frac{x}{\pi} \sin nx\, dx$$

$$= \frac{1}{\pi} \left[x \frac{-\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi$$

$$= \frac{1}{\pi} \left[-\pi \frac{\cos n\pi}{n} - 0 + \frac{\sin n\pi}{n^2} - \sin 0 \right]$$

$$= -\frac{(-1)^n}{n} = \frac{(-1)^{n+1}}{n} = F_s(n)$$

4. Find $f(x)$ if its finite fourier cosine transform is $f_c(n) = \frac{\sin \frac{n\pi}{2}}{2n}$ $n = 1, 2, 3 \ldots$

$= \frac{\pi}{4}$ if $n = 0$ and given $0 < x < 2\pi$

Sol:- From inverse finite fourier cosine transform of $F_c(n)$, is given by

$$F(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l}$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{4}\right) + \frac{2}{2\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{2n} \cos \frac{nx}{2} \quad \left[\because l = 2\pi,\ f_c(0) = \frac{\pi}{4}\right]$$

$$= \frac{1}{8} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos \frac{nx}{2}$$

5 Find $f(x)$ if $F_s(f(x)) = \frac{16(-1)^{n-1}}{n^3}$ if $0 < x < 8$ when $1, 2, 3 \ldots$

Sol:- Given $F_s(n) = \frac{16(-1)^{n-1}}{n^3}$ if $0 < x < 8$

and $l = 8$

The finite inverse finite fourier sine transform of $f(x)$ is given by

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l}$$

$$= \frac{2}{8} \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{n^3} \sin \frac{n\pi x}{8}$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin \frac{n\pi}{8} x$$

6 solve the integral equation $\int_0^{\infty} f(x) \cos \alpha x \, dx = e^{-\alpha}$ and find $f(x)$.

Sol: Given $F_c(\alpha) = e^{-\alpha}$

By inverse fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\alpha) \cos \alpha x \, d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-\alpha} \cos \alpha x \, d\alpha \quad \left[\because e^{-ax} \cos bx = \frac{a}{a^2+b^2}\right]$$

$$\frac{2}{\pi} \frac{1}{1+x^2}$$

Scanned with CamScanner

Scanned with CamScanner

Parseval's identity for fourier transform:-

If $F(\alpha)$ & $G(\alpha)$ are fourier transforms of $f(x)$ and $g(x)$ respectively, then

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,\overline{G(\alpha)}\,d\alpha = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$ where bar stands for complex conjugate

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\,d\alpha = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

Parsevals identities for fourier sine transforms.

If $F_s(\alpha)$ and $G_s(\alpha)$ are the fourier sine transform of $f(x)$ and $g(x)$ respectively, then

(i) $\frac{2}{\pi}\int_{0}^{\infty} F_s(\alpha)\,G_s(\alpha)\,d\alpha = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(ii) $\frac{2}{\pi}\int_{0}^{\infty} |F_s(\alpha)|^2\,d\alpha = \int_{0}^{\infty} |f(x)|^2\,dx$

If ~~$F_c(\alpha)$~~ Parsevals indentity for fourier cosine transforms:-

If $F_c(\alpha)$ and $G_c(\alpha)$ are the fourier cosine transforms of $f(x)$ and $g(x)$ respectively, then

(i) $\frac{2}{\pi}\int_{0}^{\infty} F_c(\alpha)\,G_c(\alpha)\,d\alpha = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(ii) $\frac{2}{\pi}\int_{0}^{\infty} |F_c(\alpha)|^2\,d\alpha = \int_{0}^{\infty} |f(x)|^2\,dx.$

1) If $f(x) = 1, \ |x|<1$ and $F(s) = \frac{2}{s}\sin s$ Using
$= 0, \ |x|>1$
parseval's identity, p.t. $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Sol:- Given $F(s) = \frac{2}{s}\sin s$

Using parseval's identity for Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2}{s}\sin s\right|^2 ds = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} 1\,dx + \int_{1}^{\infty} 0\,dx$$

$$\Rightarrow \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{4}^{2}\sin^2 s}{s^2} ds = [x]_{-1}^{1}$$

$$\Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = [1-(-1)] = 2$$

$$\Rightarrow \frac{\cancel{2}}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = \cancel{2}$$

$$\Rightarrow \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = \pi \quad \left[\frac{\sin^2 s}{s^2} \text{ is an even function of } s\right]$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin^2 s}{s^2} ds = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 s}{s^2} ds = \frac{\pi}{2} \quad (\text{or}) \quad \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2). Show that the Fourier transform of $f(x) = 1-x^2$, if $|x|<1$
$= 0$, if $|x|>1$

is $\frac{4}{p^3}(\sin p - p\cos p)$. Using parseval's identity prove that
$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$$

Sol:- For Fourier transform of $f(x)$, Refer 4th problem in (F.T'S)

i.e. $F(p) = \frac{4}{p^3}(\sin p - p\cos p)$ —— (1)

Using parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(p)|^2\,dp = \int_{-\infty}^{\infty}|f(x)|^2\,dx.$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{4(\sin p - p\cos p)}{p^3}\right]^2 dp = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1}(1-x^2)^2\,dx + \int_{1}^{\infty} 0\,dx.$$

i.e. $$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \int_{-1}^{1}(1+x^4-2x^2)\,dx$$

i.e. $$\frac{8}{\pi}\,2\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \left[x+\frac{x^5}{5}-2\frac{x^3}{3}\right]_{-1}^{1}$$

i.e. $$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \left[\left(1+\frac{1}{5}-\frac{2}{3}\right)-\left(-1-\frac{1}{5}+\frac{2}{3}\right)\right]$$

i.e. $$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = 2+\frac{2}{5}-\frac{4}{3} = \frac{30+6-20}{15} = \frac{16}{15}$$

i.e. $$\frac{\cancel{16}}{\pi}\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{\cancel{16}}{15}$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{1}{15}$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{\pi}{15}$$

(or) $$\boxed{\int_{0}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.}$$

1) Find the F.T. of $f(x)$ defined by $f(x) = 1, \ |x|<a$
$\qquad\qquad\qquad\qquad\qquad\qquad = 0, \ |x|>a$

and hence p.t. $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2} = \frac{\pi a}{2}$.

2) Find the F.T. of $f(x) = 1-|x|$ when $|x|<1$
$\qquad\qquad\qquad\qquad = 0$ when $|x|>1$

and hence deduce that $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$.

2. Using parseval's identity, Evaluate $\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx$ (an

Sol:-

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx = \int_0^\infty \frac{x}{a^2+x^2} \cdot \frac{x}{a^2+x^2} dx$$

let $f(x) = e^{-ax}$ then $F_s(\alpha) = \int_0^\infty e^{-ax} \sin\alpha x \, dx$

$$= \frac{\alpha}{a^2+\alpha^2}$$

By parsevals identity for fourier sine transform

$$\frac{2}{\pi} \int_0^\infty [F_s(\alpha)]^2 d\alpha = \int_0^\infty |f(x)|^2 dx$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\alpha}{\alpha^2+a^2}\right)^2 d\alpha = \int_0^\infty (e^{-ax})^2 dx$$

$$= \int_0^\infty e^{-2ax} dx$$

$$= \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty = \frac{-1}{2a}(0 - e^{-0})$$

$$= \frac{1}{2a}$$

Scanned with CamScanner

Scanned with CamScanner

$$\Rightarrow \int_0^\infty \left(\frac{\alpha}{\alpha^2+a^2}\right)^2 d\alpha = \frac{\pi}{4a}.$$

(or) $$\int_0^\infty \frac{a^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a}$$

3 Using parsevals identity, Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Sol: $$\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \int_0^\infty \frac{a}{a^2+x^2}\,\frac{a}{a^2+x^2}\,dx$$

let $f(x) = e^{-ax}$ then $F_c(\alpha) = \int_0^\infty e^{-ax}\cos\alpha x\,dx$

$$= \frac{a}{a^2+\alpha^2}$$

By parsevals identity for fourier transform

$$\frac{2}{\pi}\int_0^\infty |F_c(\alpha)|^2 d\alpha = \int_0^\infty |f(x)|^2 dx$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{a}{a^2+\alpha^2}\right)^2 d\alpha = \int_0^\infty (e^{-ax})^2 dx$$

$$= \int_0^\infty e^{-2ax}\,dx$$

$$= \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$= \frac{-1}{2a}(0-e^0) = \frac{1}{2a}$$

$$= \int_0^\infty \frac{a^2}{(a^2+\alpha^2)^2}\,d\alpha = \frac{\pi}{4a}$$

$$\int_0^\infty \frac{d\alpha}{(a^2+\alpha^2)^2} = \frac{\pi}{4a^3}$$

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}\ //$$

4. Using parsevals identity, show that $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2(a+b)}$

Sol: $$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \int_0^\infty \frac{x}{x^2+a^2}\,\frac{x}{x^2+b^2}\,dx$$

let $f(x) = e^{-ax}$ then $F_s(\alpha) = \int_0^\infty e^{-ax}\sin\alpha x\,dx = \frac{\alpha}{\alpha^2+a^2}$

$g(x) = e^{-bx}$ then $G_s(\alpha) = \frac{\alpha}{\alpha^2+b^2}$

By parsevals identity for Fourier sine transform

$$\frac{2}{\pi}\int_0^\infty F_s(\alpha)\, G_s(\alpha)\, d\alpha = \int_0^\infty f(x)\, g(x)\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+a^2}\,\frac{\alpha}{\alpha^2+b^2}\, d\alpha = \int_0^\infty e^{-ax}\, e^{-bx}\, dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\alpha^2}{(\alpha^2+a^2)(\alpha^2+b^2)}\, d\alpha = \int_0^\infty e^{-x(a+b)}\, dx$$

$$= \left(\frac{e^{-(a+b)x}}{-(a+b)}\right)_0^\infty$$

$$= \frac{-1}{a+b}\left(0-e^{-0}\right) = \frac{1}{a+b}.$$

$$\Rightarrow \int_0^\infty \frac{\alpha^2}{(\alpha^2+a^2)(\alpha^2+b^2)}\, d\alpha = \frac{\pi}{2(a+b)}$$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2(a+b)}$$